Exercise 15

A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 minutes?

(b) When will the turkey have cooled to 100°F?

Solution

Assume that the rate of decrease of the turkey's temperature is proportional to the difference between the turkey's temperature and the surrounding temperature T_s.

$$\frac{dT}{dt} \propto -(T - T_s)$$

The minus sign is included so that when the surroundings are cooler (hotter) than the turkey, dT/dt is negative (positive). Change this proportionality to an equation by introducing a positive constant k.

$$\frac{dT}{dt} = -k(T - T_s)$$

To solve this differential equation for T, make the substitution $y = T - T_s$.

$$\frac{dT}{dt} = -ky$$

Differentiate both sides of the substitution with respect to t to write the derivative in terms of y: $\frac{dy}{dt} = \frac{d}{dt}(T - T_s) = \frac{dT}{dt}$.

$$\frac{dy}{dt} = -ky$$

Divide both sides by y.

$$\frac{1}{y}\frac{dy}{dt} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dt}\ln y = -k$$

The function you take a derivative of to get $-k$ is $-kt + C$, where C is any constant.

$$\ln y = -kt + C$$

Exponentiate both sides to get y.

$$e^{\ln y} = e^{-kt+C}$$

$$y = e^C e^{-kt}$$

Use a new constant A for e^C.

$$y(t) = Ae^{-kt}$$

Now that the differential equation has been solved, change back to the original variable T, the turkey's temperature.

$$T - T_s = Ae^{-kt}$$

As a result,

$$T(t) = T_s + Ae^{-kt}.$$

Since the room temperature is 75°F, $T_s = 75$.

$$T(t) = 75 + Ae^{-kt}.$$

Use the fact that the turkey's initial temperature is 185°F to determine A.

$$185 = 75 + Ae^{-k(0)} \quad \rightarrow \quad A = 185 - 75 = 110$$

Consequently,

$$T(t) = 75 + 110e^{-kt}.$$

Part (a)

Use the fact that the temperature of the turkey is 150°F after half an hour, or 30 minutes, to determine k.

$$150 = 75 + 110e^{-k(30)}$$

$$75 = 110e^{-30k}$$

$$\frac{75}{110} = e^{-30k}$$

$$\frac{15}{22} = e^{-30k}$$

$$\ln \frac{15}{22} = \ln e^{-30k}$$

$$\ln \frac{15}{22} = (-30k) \ln e$$

$$k = -\frac{\ln \frac{15}{22}}{30} \approx 0.0127664 \text{ day}^{-1}$$

Therefore, the turkey's temperature is

$$\begin{aligned} T(t) &= 75 + 110e^{-kt} \\ &= 75 + 110e^{-\left(-\frac{\ln \frac{15}{22}}{30}\right)t} \\ &= 75 + 110e^{\ln\left(\frac{15}{22}\right)^{t/30}} \\ &= 75 + 110\left(\frac{15}{22}\right)^{t/30}, \end{aligned}$$

and the temperature after 45 minutes is

$$T(45) = 75 + 110\left(\frac{15}{22}\right)^{45/30} \approx 136.929°\text{F}.$$

Part (b)

To find when the turkey will have cooled to 100°F, set $T(t) = 100$ and solve the equation for t.

$$T(t) = 100$$

$$75 + 110\left(\frac{15}{22}\right)^{t/30} = 100$$

$$110\left(\frac{15}{22}\right)^{t/30} = 25$$

$$\left(\frac{15}{22}\right)^{t/30} = \frac{25}{110}$$

$$\ln\left(\frac{15}{22}\right)^{t/30} = \ln\frac{25}{110}$$

$$\left(\frac{t}{30}\right)\ln\frac{15}{22} = \ln\frac{5}{22}$$

$$t = \frac{30\ln\frac{5}{22}}{\ln\frac{15}{22}} \approx 116.055 \text{ minutes}$$