Problem 2

Evaluate

$$\int \frac{1}{x^7 - x}\,dx$$

The straightforward approach would be to start with partial fractions, but that would be brutal. Try a substitution.

Solution

Following the suggestion, we will look for a substitution. Factor out x from the denominator.

$$\int \frac{1}{x(x^6 - 1)}\,dx$$

Make the following u-substitution.

$$\begin{aligned} u &= x^6 - 1 \quad &\rightarrow \quad u + 1 &= x^6 \\ du &= 6x^5\,dx \quad &\rightarrow \quad \frac{du}{6x^5} &= dx \end{aligned}$$

Then the integral becomes

$$\int \frac{1}{xu}\frac{du}{6x^5}.$$

Bring the constant out in front and combine the x-terms.

$$\frac{1}{6}\int \frac{1}{ux^6}\,du$$

Substitute the expression for x^6.

$$\frac{1}{6}\int \frac{1}{u(u+1)}\,du$$

Split up the integrand with partial fraction decomposition.

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

Multiply both sides by the least common denominator.

$$1 = A(u+1) + Bu$$

As we have two unknowns, A and B, choose two random values of u to get two equations to solve for them.

$$\begin{aligned} u = 0: &\quad 1 = A(1) \\ u = 1: &\quad 1 = A(2) + B \end{aligned}$$

Solving the system yields $A = 1$ and $B = -1$, so we have the following for the integral.

$$\frac{1}{6}\int \left(\frac{1}{u} - \frac{1}{u+1}\right) du$$

Split up the integral into two.

$$\frac{1}{6}\left(\int \frac{1}{u}\,du - \int \frac{1}{u+1}\,du\right)$$

Now integrate the two functions.

$$\frac{1}{6}(\ln|u| - \ln|u+1|) + C,$$

where C is an arbitrary constant. Since the integral we have to solve is in terms of x, that's what the final answer should be in terms of.

$$\frac{1}{6}\ln|x^6 - 1| - \frac{1}{6}\ln|x^6| + C$$

Therefore,

$$\int \frac{1}{x^7 - x}\,dx = \frac{1}{6}\ln|x^6 - 1| - \ln|x| + C.$$