Exercise 35

An **integral equation** is an equation that contains an unknown function $y(x)$ and an integral that involves $y(x)$. Solve the given integral equation. [*Hint:* Use an initial condition obtained from the integral equation.]

$$y(x) = 4 + \int_0^x 2t\sqrt{y(t)}\,dt$$

Solution

Right away we see that if we plug in $x = 0$ to the integral equation, we get

$$\begin{aligned} y(0) &= 4 + \int_0^0 2t\sqrt{y(t)}\,dt \\ &= 4 + 0 \\ &= 4, \end{aligned}$$

so the initial condition is $y(0) = 4$. In order to solve for $y(x)$ by the method introduced in this section, differentiate both sides of the integral equation with respect to x.

$$\frac{d}{dx}y(x) = \underbrace{\frac{d}{dx}4}_{=\,0} + \frac{d}{dx}\int_0^x 2t\sqrt{y(t)}\,dt$$

By the fundamental theorem of calculus,

$$\frac{d}{dx}\int_0^x 2t\sqrt{y(t)}\,dt = 2x\sqrt{y(x)},$$

so the differential equation we need to solve is the following.

$$\frac{dy}{dx} = 2x\sqrt{y}$$

This is a separable equation, which means we can solve for $y(x)$ by bringing all terms with y to the left and all constants and terms with x to the right and then integrating both sides.

$$\begin{aligned} dy &= 2x\sqrt{y}\,dx \\ \frac{dy}{\sqrt{y}} &= 2x\,dx \\ \int \frac{dy}{\sqrt{y}} &= \int 2x\,dx \\ \frac{1}{\frac{1}{2}}\sqrt{y} &= x^2 + C \\ \sqrt{y} &= \frac{1}{2}x^2 + \frac{1}{2}C \end{aligned}$$

Let $C_1 = \frac{1}{2}C$. Then

$$y(x) = \left(\frac{1}{2}x^2 + C_1\right)^2.$$

Now we use the initial condition, $y(0) = 4$ to determine C_1.

$$\begin{aligned} y(0) = (C_1)^2 &= 4 \\ C_1 &= \pm 2 \end{aligned}$$

If we plug in $C_1 = -2$ into $y(x)$, it doesn't satisfy the integral equation, so we have to use $C_1 = 2$. Therefore,

$$y(x) = \left(\frac{1}{2}x^2 + 2\right)^2.$$