Exercise 7

Are the functions $1 + x$, $1 - x$, and $1 + x + x^2$ linearly dependent or independent? Why?

Solution

To determine whether a set of functions $\mathcal{S} = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ is linearly independent or not, one must consider the Wronski matrix.

$$\mathbf{W}(x) = \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}$$

If there is a point $x = x_0$ such that $\mathbf{W}(x_0)$ is nonsingular (i.e. the determinant is nonzero), then the set of functions is linearly independent.

The associated Wronski matrix for the set of functions in this exercise is

$$\mathbf{W}(x) = \begin{bmatrix} 1+x & 1-x & 1+x+x^2 \\ 1 & -1 & 1+2x \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det \mathbf{W}(x) &= \begin{vmatrix} 1+x & 1-x & 1+x+x^2 \\ 1 & -1 & 1+2x \\ 0 & 0 & 2 \end{vmatrix} \\ &= (1+x)\begin{vmatrix} -1 & 1+2x \\ 0 & 2 \end{vmatrix} - (1-x)\begin{vmatrix} 1 & 1+2x \\ 0 & 2 \end{vmatrix} + (1+x+x^2)\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} \\ &= (1+x)(-2) - (1-x)(2) + (1+x+x^2)(0) \\ &= -4 \end{aligned}$$

The determinant of $\mathbf{W}(x)$ is nonzero for all values of x, so $\mathbf{W}(x)$ is nonsingular. Therefore, the set of functions $\{1 + x, 1 - x, 1 + x + x^2\}$ is linearly independent.