Exercise 2

Which of the following operators are linear?

(a) $\mathscr{L}u = u_x + xu_y$

(b) $\mathscr{L}u = u_x + uu_y$

(c) $\mathscr{L}u = u_x + u_y^2$

(d) $\mathscr{L}u = u_x + u_y + 1$

(e) $\mathscr{L}u = \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$

Solution

To determine if an operator is linear, one must check whether the conditions for linearity hold:

$$\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v \quad \text{and} \quad \mathscr{L}(cu) = c\mathscr{L}u$$

Part (a)

$$\begin{aligned}
\mathscr{L}u &= u_x + xu_y \\
\mathscr{L}u &= \frac{\partial}{\partial x}u + x\frac{\partial}{\partial y}u \\
&= \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)u
\end{aligned}$$

Therefore,

$$\mathscr{L} = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)(u+v) \\
&= \frac{\partial}{\partial x}(u+v) + x\frac{\partial}{\partial y}(u+v) \\
&= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + x\frac{\partial}{\partial y}u + x\frac{\partial}{\partial y}v \\
&= \frac{\partial}{\partial x}u + x\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v + x\frac{\partial}{\partial y}v \\
&= \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)u + \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

The first condition holds. Now the second one must be checked.

$$\begin{aligned}
\mathscr{L}(cu) &= \left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)(cu) \\
&= \frac{\partial}{\partial x}(cu) + x\frac{\partial}{\partial y}(cu) \\
&= c\frac{\partial}{\partial x}u + cx\frac{\partial}{\partial y}u \\
&= c\left(\frac{\partial}{\partial x}u + x\frac{\partial}{\partial y}u\right) \\
&= c\left(\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)u \\
&= c\mathscr{L}u
\end{aligned}$$

The second condition holds as well, so the operator is linear.

Part (b)

$$\begin{aligned}
\mathscr{L}u &= u_x + uu_y \\
\mathscr{L}u &= \frac{\partial}{\partial x}u + u\frac{\partial}{\partial y}u \\
&= \left(\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right)u
\end{aligned}$$

Therefore,

$$\mathscr{L} = \frac{\partial}{\partial x} + u\frac{\partial}{\partial y}$$

$$\begin{aligned}
\mathscr{L}(u+v) &= \left[\frac{\partial}{\partial x} + (u+v)\frac{\partial}{\partial y}\right](u+v) \\
&= \frac{\partial}{\partial x}(u+v) + (u+v)\frac{\partial}{\partial y}(u+v) \\
&= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + (u+v)\left(\frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v\right) \\
&= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + u\frac{\partial}{\partial y}u + u\frac{\partial}{\partial y}v + v\frac{\partial}{\partial y}u + v\frac{\partial}{\partial y}v \\
&= \frac{\partial}{\partial x}u + u\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + u\frac{\partial}{\partial y}v + v\frac{\partial}{\partial y}u \\
&= \left(\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right)u + \left(\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)v + u\frac{\partial}{\partial y}v + v\frac{\partial}{\partial y}u \\
&= \mathscr{L}u + \mathscr{L}v + uv_y + vu_y
\end{aligned}$$

The first condition does not hold, so the PDE is nonlinear.

Part (c)

$$\begin{aligned}\mathscr{L}u &= u_x + u_y^2 \\ \mathscr{L}u &= \frac{\partial}{\partial x}u + \left(\frac{\partial}{\partial y}u\right)^2 \\ &= \left[\frac{\partial}{\partial x} + \left(\frac{\partial}{\partial y}u\right)\frac{\partial}{\partial y}\right]u\end{aligned}$$

Therefore,

$$\mathscr{L} = \frac{\partial}{\partial x} + \left(\frac{\partial}{\partial y}u\right)\frac{\partial}{\partial y}$$

$$\begin{aligned}\mathscr{L}(u+v) &= \left\{\frac{\partial}{\partial x} + \left[\frac{\partial}{\partial y}(u+v)\right]\frac{\partial}{\partial y}\right\}(u+v) \\ &= \frac{\partial}{\partial x}(u+v) + \left[\frac{\partial}{\partial y}(u+v)\right]\frac{\partial}{\partial y}(u+v) \\ &= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + \left(\frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v\right)\left(\frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v\right) \\ &= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + \left(\frac{\partial}{\partial y}u\right)^2 + \left(\frac{\partial}{\partial y}u\right)\left(\frac{\partial}{\partial y}v\right) + \left(\frac{\partial}{\partial y}v\right)\left(\frac{\partial}{\partial y}u\right) + \left(\frac{\partial}{\partial y}v\right)^2 \\ &= \frac{\partial}{\partial x}u + \left(\frac{\partial}{\partial y}u\right)^2 + \frac{\partial}{\partial x}v + \left(\frac{\partial}{\partial y}v\right)^2 + 2\left(\frac{\partial}{\partial y}u\right)\left(\frac{\partial}{\partial y}v\right) \\ &= \left[\frac{\partial}{\partial x} + \left(\frac{\partial}{\partial y}u\right)\frac{\partial}{\partial y}\right]u + \left[\frac{\partial}{\partial x} + \left(\frac{\partial}{\partial y}v\right)\frac{\partial}{\partial y}\right]v + 2\left(\frac{\partial}{\partial y}u\right)\left(\frac{\partial}{\partial y}v\right) \\ &= \mathscr{L}u + \mathscr{L}v + 2u_y v_y\end{aligned}$$

The first condition does not hold, so the PDE is nonlinear.

Part (d)

$$\begin{aligned}\mathscr{L}u &= u_x + u_y + 1 \\ \mathscr{L}u &= \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}u + 1 \\ &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{1}{u}\right)u\end{aligned}$$

Therefore,

$$\mathscr{L} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{1}{u}$$

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{1}{u+v}\right)(u+v) \\
&= \frac{\partial}{\partial x}(u+v) + \frac{\partial}{\partial y}(u+v) + 1 \\
&= \frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v + \frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v + 1 \\
&= \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}u + 1 + \frac{\partial}{\partial x}v + \frac{\partial}{\partial y}v + 1 - 1 \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{1}{u}\right)u + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{1}{v}\right)v - 1 \\
&= \mathscr{L}u + \mathscr{L}v - 1
\end{aligned}$$

The first condition does not hold, so the PDE is nonlinear.

Part (e)

$$\begin{aligned}
\mathscr{L}u &= \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - [\arctan{(x/y)}]u \\
\mathscr{L}u &= \sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x}u + \frac{\partial^3}{\partial y\partial x\partial y}u - \left(\arctan\frac{x}{y}\right)u \\
&= \left(\sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x} + \frac{\partial^3}{\partial y\partial x\partial y} - \arctan\frac{x}{y}\right)u
\end{aligned}$$

Therefore,

$$\mathscr{L} = \sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x} + \frac{\partial^3}{\partial y\partial x\partial y} - \arctan\frac{x}{y}$$

$$\begin{aligned}
\mathscr{L}(u+v) &= \left(\sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x} + \frac{\partial^3}{\partial y\partial x\partial y} - \arctan\frac{x}{y}\right)(u+v) \\
&= \sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x}(u+v) + \frac{\partial^3}{\partial y\partial x\partial y}(u+v) - \left(\arctan\frac{x}{y}\right)(u+v) \\
&= \sqrt{1+x^2}(\cos y)\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial x}v\right) + \left(\frac{\partial^3}{\partial y\partial x\partial y}u + \frac{\partial^3}{\partial y\partial x\partial y}v\right) - \left(\arctan\frac{x}{y}\right)(u+v) \\
&= \left(\sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x} + \frac{\partial^3}{\partial y\partial x\partial y} - \arctan\frac{x}{y}\right)u \\
&\qquad + \left(\sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x} + \frac{\partial^3}{\partial y\partial x\partial y} - \arctan\frac{x}{y}\right)v \\
&= \mathscr{L}u + \mathscr{L}v
\end{aligned}$$

The first condition holds. Now the second one must be checked.

$$
\begin{aligned}
\mathscr{L}(cu) &= \left(\sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x} + \frac{\partial^3}{\partial y \partial x \partial y} - \arctan\frac{x}{y}\right)(cu) \\
&= \sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x}(cu) + \frac{\partial^3}{\partial y \partial x \partial y}(cu) - \left(\arctan\frac{x}{y}\right)(cu) \\
&= c\sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x}u + c\frac{\partial^3}{\partial y \partial x \partial y}u - c\left(\arctan\frac{x}{y}\right)u \\
&= c\left[\sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x}u + \frac{\partial^3}{\partial y \partial x \partial y}u - \left(\arctan\frac{x}{y}\right)u\right] \\
&= c\left(\sqrt{1+x^2}(\cos y)\frac{\partial}{\partial x} + \frac{\partial^3}{\partial y \partial x \partial y} - \arctan\frac{x}{y}\right)u \\
&= c\mathscr{L}u
\end{aligned}
$$

The second condition holds as well, so the operator is linear.

To summarize, (a) and (e) are linear, whereas (b), (c), and (d) are nonlinear.