Exercise 4

Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same g is a solution of the homogeneous equation $\mathscr{L}u = 0$.

Solution

Suppose that v and w are solutions to this inhomogeneous linear equation.

$$\mathscr{L}v = g \quad \text{and} \quad \mathscr{L}w = g$$

Since the equation is linear, the operator $\mathscr{L}$ satisfies the two conditions for linearity, namely $\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v$ and $\mathscr{L}(cu) = c\mathscr{L}u$. Subtracting the second equation from the first, we get

$$\begin{aligned} \mathscr{L}v - \mathscr{L}w &= g - g \\ \mathscr{L}v + \mathscr{L}(-w) &= 0 \\ \mathscr{L}[v + (-w)] &= 0 \\ \mathscr{L}(v - w) &= 0 \end{aligned}$$

Therefore, the difference of two solutions of an inhomogeneous linear equation is a solution of the homogeneous equation.