Exercise 5

Which of the following collections of 3-vectors $[a, b, c]$ are vector spaces? Provide reasons.

(a) The vectors with $b = 0$.

(b) The vectors with $b = 1$.

(c) The vectors with $ab = 0$.

(d) All the linear combinations of the two vectors $[1, 1, 0]$ and $[2, 0, 1]$.

(e) All the vectors such that $c - a = 2b$.

Solution

For a collection of 3-vectors $\mathcal{V}$ to be a vector space over $\mathbb{R}$, the vector addition and scalar multiplication operations must satisfy the following ten properties:

(A1) $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A2) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{V}$.

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathbb{R}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for every $\alpha \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for all $\alpha, \beta \in \mathbb{R}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

Part (a)

$$\mathcal{V} = \{[a, 0, c] \mid a, c \in \mathbb{R}\}$$

Choose $\mathbf{x} = [a_1, 0, c_1] \in \mathcal{V}$ and $\mathbf{y} = [a_2, 0, c_2] \in \mathcal{V}$ and $\mathbf{z} = [a_3, 0, c_3] \in \mathcal{V}$.

$$\mathbf{x} + \mathbf{y} = [a_1, 0, c_1] + [a_2, 0, c_2] = [a_1 + a_2, 0, c_1 + c_2]$$

Because $a_1 + a_2 \in \mathbb{R}$ and $c_1 + c_2 \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} \in \mathcal{V}$. Hence, property A1 is satisfied.

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= ([a_1, 0, c_1] + [a_2, 0, c_2]) + [a_3, 0, c_3] \\
&= [a_1 + a_2, 0, c_1 + c_2] + [a_3, 0, c_3] \\
&= [a_1 + a_2 + a_3, 0, c_1 + c_2 + c_3] \\
&= [a_1, 0, c_1] + [a_2 + a_3, 0, c_2 + c_3] \\
&= [a_1, 0, c_1] + ([a_2, 0, c_2] + [a_3, 0, c_3]) \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z})
\end{aligned}$$

Property A2 is satisfied.

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= [a_1, 0, c_1] + [a_2, 0, c_2] \\ &= [a_1 + a_2, 0, c_1 + c_2] \\ &= [a_2 + a_1, 0, c_2 + c_1] \\ &= [a_2, 0, c_2] + [a_1, 0, c_1] \\ &= \mathbf{y} + \mathbf{x}\end{aligned}$$

Property A3 is satisfied. Setting $a = 0$ and $c = 0$, we find that $\mathbf{0} = [0, 0, 0] \in \mathcal{V}$, so property A4 is satisfied. Since $-a_1$ and $-c_1$ are members of $\mathbb{R}$, $-\mathbf{x} \in \mathcal{V}$, and property A5 is satisfied.
Choose $\alpha_1 \in \mathbb{R}$ and $\beta_1 \in \mathbb{R}$.

$$\begin{aligned}\alpha_1\mathbf{x} &= \alpha_1[a_1, 0, c_1] \\ &= [\alpha_1 a_1, 0, \alpha_1 c_1]\end{aligned}$$

Since $\alpha_1 a_1$ and $\alpha_1 c_1$ are members of $\mathbb{R}$, $\alpha_1\mathbf{x} \in \mathcal{V}$, and property M1 is satisfied.

$$\begin{aligned}(\alpha_1\beta_1)\mathbf{x} &= (\alpha_1\beta_1)[a_1, 0, c_1] \\ &= [\alpha_1\beta_1 a_1, 0, \alpha_1\beta_1 c_1] \\ &= \alpha_1[\beta_1 a_1, 0, \beta_1 c_1] \\ &= \alpha_1(\beta_1\mathbf{x})\end{aligned}$$

So property M2 is satisfied.

$$\begin{aligned}\alpha_1(\mathbf{x}+\mathbf{y}) &= \alpha_1([a_1, 0, c_1] + [a_2, 0, c_2]) \\ &= \alpha_1[a_1 + a_2, 0, c_1 + c_2] \\ &= [\alpha_1(a_1 + a_2), 0, \alpha_1(c_1 + c_2)] \\ &= [\alpha_1 a_1 + \alpha_1 a_2, 0, \alpha_1 c_1 + \alpha_1 c_2] \\ &= [\alpha_1 a_1, 0, \alpha_1 c_1] + [\alpha_1 a_2, 0, \alpha_1 c_2] \\ &= \alpha_1[a_1, 0, c_1] + \alpha_1[a_2, 0, c_2] \\ &= \alpha_1\mathbf{x} + \alpha_1\mathbf{y}\end{aligned}$$

So property M3 is satisfied.

$$\begin{aligned}(\alpha_1+\beta_1)\mathbf{x} &= (\alpha_1+\beta_1)[a_1, 0, c_1] \\ &= [(\alpha_1+\beta_1)a_1, 0, (\alpha_1+\beta_1)c_1] \\ &= [\alpha_1 a_1 + \beta_1 a_1, 0, \alpha_1 c_1 + \beta_1 c_1] \\ &= [\alpha_1 a_1, 0, \alpha_1 c_1] + [\beta_1 a_1, 0, \beta_1 c_1] \\ &= \alpha_1[a_1, 0, c_1] + \beta_1[a_1, 0, c_1] \\ &= \alpha_1\mathbf{x} + \beta_1\mathbf{x}\end{aligned}$$

So property M4 is satisfied. $1\mathbf{x} = 1[a_1, 0, c_1] = [1 \times a_1, 1 \times 0, 1 \times c_1] = [a_1, 0, c_1] = \mathbf{x}$, so property M5 is satisfied.

All ten properties are satisfied, so $\mathcal{V} = \{[a, 0, c] \mid a, c \in \mathbb{R}\}$ is a vector space.

Part (b)

$$\mathcal{V} = \{[a, 1, c] \mid a, c \in \mathbb{R}\}$$

There is no $\mathbf{0} = [0, 0, 0]$ vector in this set of 3-vectors, so property A4 is not satisfied. Therefore, $\mathcal{V} = \{[a, 1, c] \mid a, c \in \mathbb{R}\}$ is not a vector space.

Part (c)

$$\mathcal{V} = \{[a, b, c] \mid ab = 0 \text{ and } a, b, c \in \mathbb{R}\}$$

Choose $\mathbf{x} = [a_1, b_1, c_1] \in \mathcal{V}$ and $\mathbf{y} = [a_2, b_2, c_2] \in \mathcal{V}$ and $\mathbf{z} = [a_3, b_3, c_3] \in \mathcal{V}$.

$$\mathbf{x} + \mathbf{y} = [a_1, b_1, c_1] + [a_2, b_2, c_2] = [a_1 + a_2, b_1 + b_2, c_1 + c_2]$$

The conditions $a_1 b_1 = 0$ and $a_2 b_2 = 0$ do not guarantee that $(a_1 + a_2)(b_1 + b_2) = 0$; for example, take $a_1 = 0$, $b_1 = 1$, $a_2 = 1$, and $b_2 = 0$. Therefore, $\mathbf{x} + \mathbf{y} \notin \mathcal{V}$, and
$\mathcal{V} = \{[a, b, c] \mid ab = 0 \text{ and } a, b, c \in \mathbb{R}\}$ is not a vector space.

Part (d)

$$\mathcal{V} = \{[a, b, c] \mid m[1, 1, 0] + n[2, 0, 1] = [a, b, c] = [m + 2n, m, n],\ m, n \in \mathbb{R}\}$$

Choose $\mathbf{x} = [m_1 + 2n_1, m_1, n_1] \in \mathcal{V}$ and $\mathbf{y} = [m_2 + 2n_2, m_2, n_2] \in \mathcal{V}$ and $\mathbf{z} = [m_3 + 2n_3, m_3, n_3] \in \mathcal{V}$.

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= [m_1 + 2n_1, m_1, n_1] + [m_2 + 2n_2, m_2, n_2] \\
&= [m_1 + 2n_1 + m_2 + 2n_2, m_1 + m_2, n_1 + n_2] \\
&= [\underbrace{(m_1 + m_2)}_{m_4} + 2\underbrace{(n_1 + n_2)}_{n_4}, m_1 + m_2, n_1 + n_2] \\
&= [m_4 + 2n_4, m_4, n_4]
\end{aligned}$$

Because $m_4 \in \mathbb{R}$ and $n_4 \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} \in \mathcal{V}$. Hence, property A1 is satisfied.

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= ([m_1 + 2n_1, m_1, n_1] + [m_2 + 2n_2, m_2, n_2]) + [m_3 + 2n_3, m_3, n_3] \\
&= [m_1 + 2n_1 + m_2 + 2n_2, m_1 + m_2, n_1 + n_2] + [m_3 + 2n_3, m_3, n_3] \\
&= [m_1 + 2n_1 + m_2 + 2n_2 + m_3 + 2n_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3] \\
&= [m_1 + 2n_1, m_1, n_1] + [m_2 + 2n_2 + m_3 + 2n_3, m_2 + m_3, n_2 + n_3] \\
&= [m_1 + 2n_1, m_1, n_1] + ([m_2 + 2n_2, m_2, n_2] + [m_3 + 2n_3, m_3, n_3]) \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z})
\end{aligned}$$

Property A2 is satisfied.

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= [m_1 + 2n_1, m_1, n_1] + [m_2 + 2n_2, m_2, n_2] \\
&= [m_1 + 2n_1 + m_2 + 2n_2, m_1 + m_2, n_1 + n_2] \\
&= [m_2 + 2n_2 + m_1 + 2n_1, m_2 + m_1, n_2 + n_1] \\
&= [m_2 + 2n_2, m_2, n_2] + [m_1 + 2n_1, m_1, n_1] \\
&= \mathbf{y} + \mathbf{x}
\end{aligned}$$

Property A3 is satisfied. Setting $m = 0$ and $n = 0$, we find that $\mathbf{0} = [0, 0, 0] \in \mathcal{V}$, so property A4 is satisfied. Since $-m_1$ and $-n_1$ and $-(m_1 + 2n_1)$ are members of $\mathbb{R}$, $-\mathbf{x} \in \mathcal{V}$, and property A5 is satisfied.
Choose $\alpha_1 \in \mathbb{R}$ and $\beta_1 \in \mathbb{R}$.

$$\begin{aligned}\alpha_1 \mathbf{x} &= \alpha_1 [m_1 + 2n_1, m_1, n_1] \\ &= [\alpha_1(m_1 + 2n_1), \alpha_1 m_1, \alpha_1 n_1]\end{aligned}$$

Since $\alpha_1(m_1 + 2n_1)$ and $\alpha_1 m_1$ and $\alpha_1 n_1$ are members of $\mathbb{R}$, $\alpha_1 \mathbf{x} \in \mathcal{V}$, and property M1 is satisfied.

$$\begin{aligned}(\alpha_1\beta_1)\mathbf{x} &= (\alpha_1\beta_1)[m_1 + 2n_1, m_1, n_1] \\ &= [\alpha_1\beta_1(m_1 + 2n_1), \alpha_1\beta_1 m_1, \alpha_1\beta_1 n_1] \\ &= \alpha_1[\beta_1(m_1 + 2n_1), \beta_1 m_1, \beta_1 n_1] \\ &= \alpha_1(\beta_1 \mathbf{x})\end{aligned}$$

So property M2 is satisfied.

$$\begin{aligned}\alpha_1(\mathbf{x} + \mathbf{y}) &= \alpha_1([m_1 + 2n_1, m_1, n_1] + [m_2 + 2n_2, m_2, n_2]) \\ &= \alpha_1[m_1 + 2n_1 + m_2 + 2n_2, m_1 + m_2, n_1 + n_2] \\ &= [\alpha_1(m_1 + 2n_1 + m_2 + 2n_2), \alpha_1(m_1 + m_2), \alpha_1(n_1 + n_2)] \\ &= [\alpha_1(m_1 + 2n_1) + \alpha_1(m_2 + 2n_2), \alpha_1 m_1 + \alpha_1 m_2, \alpha_1 n_1 + \alpha_1 n_2] \\ &= [\alpha_1(m_1 + 2n_1), \alpha_1 m_1, \alpha_1 n_1] + [\alpha_1(m_2 + 2n_2), \alpha_1 m_2, \alpha_1 n_2] \\ &= \alpha_1[m_1 + 2n_1, m_1, n_1] + \alpha_1[m_2 + 2n_2, m_2, n_2] \\ &= \alpha_1 \mathbf{x} + \alpha_1 \mathbf{y}\end{aligned}$$

So property M3 is satisfied.

$$\begin{aligned}(\alpha_1 + \beta_1)\mathbf{x} &= (\alpha_1 + \beta_1)[m_1 + 2n_1, m_1, n_1] \\ &= [(\alpha_1 + \beta_1)(m_1 + 2n_1), (\alpha_1 + \beta_1)m_1, (\alpha_1 + \beta_1)n_1] \\ &= [\alpha_1(m_1 + 2n_1) + \beta_1(m_1 + 2n_1), \alpha_1 m_1 + \beta_1 m_1, \alpha_1 n_1 + \beta_1 n_1] \\ &= [\alpha_1(m_1 + 2n_1), \alpha_1 m_1, \alpha_1 n_1] + [\beta_1(m_1 + 2n_1), \beta_1 m_1, \beta_1 n_1] \\ &= \alpha_1[m_1 + 2n_1, m_1, n_1] + \beta_1[m_1 + 2n_1, m_1, n_1] \\ &= \alpha_1 \mathbf{x} + \beta_1 \mathbf{x}\end{aligned}$$

So property M4 is satisfied.
$1\mathbf{x} = 1[m_1 + 2n_1, m_1, n_1] = [1(m_1 + 2n_1), 1 \times m_1, 1 \times n_1] = [m_1 + 2n_1, m_1, n_1] = \mathbf{x}$, so property M5 is satisfied.

All ten properties are satisfied, so $\mathcal{V} = \{[a, b, c] \mid m[1, 1, 0] + n[2, 0, 1] = [a, b, c]\}$ is a vector space.

Part (e)

$$\mathcal{V} = \{[a, b, c] \mid c - a = 2b,\ a, c \in \mathbb{R}\}$$

Choose $\mathbf{x} = [a_1, \frac{1}{2}(c_1 - a_1), c_1] \in \mathcal{V}$ and $\mathbf{y} = [a_2, \frac{1}{2}(c_2 - a_2), c_2] \in \mathcal{V}$ and $\mathbf{z} = [a_3, \frac{1}{2}(c_3 - a_3), c_3] \in \mathcal{V}$.

$$\begin{aligned}\mathbf{x} + \mathbf{y} = \left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] + \left[a_2, \frac{1}{2}(c_2 - a_2), c_2\right] &= \left[\underbrace{a_1 + a_2}_{a_4}, \frac{1}{2}\{(c_1 + c_2) - (a_1 + a_2)\}, \underbrace{c_1 + c_2}_{c_4}\right] \\ &= \left[a_4, \frac{1}{2}(c_4 - a_4), c_4\right]\end{aligned}$$

Because $a_4 \in \mathbb{R}$ and $c_4 \in \mathbb{R}$ and the second component is $\frac{1}{2}(c_4 - a_4)$, $\mathbf{x} + \mathbf{y} \in \mathcal{V}$. Hence, property A1 is satisfied.

$$\begin{aligned}(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= \left(\left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] + \left[a_2, \frac{1}{2}(c_2 - a_2), c_2\right]\right) + \left[a_3, \frac{1}{2}(c_3 - a_3), c_3\right] \\ &= \left[a_1 + a_2, \frac{1}{2}\{(c_1 + c_2) - (a_1 + a_2)\}, c_1 + c_2\right] + \left[a_3, \frac{1}{2}(c_3 - a_3), c_3\right] \\ &= \left[a_1 + a_2 + a_3, \frac{1}{2}\{(c_1 + c_2 + c_3) - (a_1 + a_2 + a_3)\}, c_1 + c_2 + c_3\right] \\ &= \left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] + \left[a_2 + a_3, \frac{1}{2}\{(c_2 + c_3) - (a_2 + a_3), c_2 + c_3\right] \\ &= \left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] + \left(\left[a_2, \frac{1}{2}(c_2 - a_2), c_2\right] + \left[a_3, \frac{1}{2}(c_3 - a_3), c_3\right]\right) \\ &= \mathbf{x} + (\mathbf{y} + \mathbf{z})\end{aligned}$$

Property A2 is satisfied.

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= \left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] + \left[a_2, \frac{1}{2}(c_2 - a_2), c_2\right] \\ &= \left[a_1 + a_2, \frac{1}{2}\{(c_1 + c_2) - (a_1 + a_2)\}, c_1 + c_2\right] \\ &= \left[a_2 + a_1, \frac{1}{2}\{(c_2 + c_1) - (a_2 + a_1)\}, c_2 + c_1\right] \\ &= \left[a_2, \frac{1}{2}(c_2 - a_2), c_2\right] + \left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] \\ &= \mathbf{y} + \mathbf{x}\end{aligned}$$

Property A3 is satisfied. Setting $a = 0$ and $c = 0$, we find that $\mathbf{0} = [0, 0, 0] \in \mathcal{V}$, so property A4 is satisfied. Since $-a_1$ and $-c_1$ are members of $\mathbb{R}$, $-\mathbf{x} \in \mathcal{V}$, and property A5 is satisfied.
Choose $\alpha_1 \in \mathbb{R}$ and $\beta_1 \in \mathbb{R}$.

$$\begin{aligned}\alpha_1\mathbf{x} &= \alpha_1 \left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] \\ &= \left[\alpha_1 a_1, \frac{1}{2}\alpha_1(c_1 - a_1), \alpha_1 c_1\right]\end{aligned}$$

Since $\alpha_1 a_1$ and $\alpha_1 c_1$ and $\frac{1}{2}\alpha_1(c_1 - a_1)$ are members of $\mathbb{R}$, $\alpha_1\mathbf{x} \in \mathcal{V}$, and property M1 is satisfied.

$$\begin{aligned}
(\alpha_1\beta_1)\mathbf{x} &= (\alpha_1\beta_1)\left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] \\
&= \left[\alpha_1\beta_1 a_1, \frac{1}{2}\alpha_1\beta_1(c_1 - a_1), \alpha_1\beta_1 c_1\right] \\
&= \alpha_1\left[\beta_1 a_1, \frac{1}{2}\beta_1(c_1 - a_1), \beta_1 c_1\right] \\
&= \alpha_1(\beta_1\mathbf{x})
\end{aligned}$$

So property M2 is satisfied.

$$\begin{aligned}
\alpha_1(\mathbf{x}+\mathbf{y}) &= \alpha_1\left(\left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] + \left[a_2, \frac{1}{2}(c_2 - a_2), c_2\right]\right) \\
&= \alpha_1\left[a_1 + a_2, \frac{1}{2}\{(c_1 + c_2) - (a_1 + a_2)\}, c_1 + c_2\right] \\
&= \left[\alpha_1(a_1 + a_2), \frac{1}{2}\alpha_1\{(c_1 + c_2) - (a_1 + a_2)\}, \alpha_1(c_1 + c_2)\right] \\
&= \left[\alpha_1 a_1 + \alpha_1 a_2, \frac{1}{2}\alpha_1(c_1 - a_1) + \frac{1}{2}\alpha_1(c_2 - a_2), \alpha_1 c_1 + \alpha_1 c_2\right] \\
&= \left[\alpha_1 a_1, \frac{1}{2}\alpha_1(c_1 - a_1), \alpha_1 c_1\right] + \left[\alpha_1 a_2, \frac{1}{2}\alpha_1(c_2 - a_2), \alpha_1 c_2\right] \\
&= \alpha_1\left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] + \alpha_1\left[a_2, \frac{1}{2}(c_2 - a_2), c_2\right] \\
&= \alpha_1\mathbf{x} + \alpha_1\mathbf{y}
\end{aligned}$$

So property M3 is satisfied.

$$\begin{aligned}
(\alpha_1 + \beta_1)\mathbf{x} &= (\alpha_1 + \beta_1)\left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] \\
&= \left[(\alpha_1 + \beta_1)a_1, \frac{1}{2}(\alpha_1 + \beta_1)(c_1 - a_1), (\alpha_1 + \beta_1)c_1\right] \\
&= \left[\alpha_1 a_1 + \beta_1 a_1, \frac{1}{2}\alpha_1(c_1 - a_1) + \frac{1}{2}\beta_1(c_1 - a_1), \alpha_1 c_1 + \beta_1 c_1\right] \\
&= \left[\alpha_1 a_1, \frac{1}{2}\alpha_1(c_1 - a_1), \alpha_1 c_1] + [\beta_1 a_1, \frac{1}{2}\beta_1(c_1 - a_1), \beta_1 c_1\right] \\
&= \alpha_1\left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] + \beta_1\left[a_1, \frac{1}{2}(c_1 - a_1), c_1\right] \\
&= \alpha_1\mathbf{x} + \beta_1\mathbf{x}
\end{aligned}$$

So property M4 is satisfied.
$1\mathbf{x} = 1[a_1, \frac{1}{2}(c_1 - a_1), c_1] = [1 \times a_1, 1 \times \frac{1}{2}(c_1 - a_1), 1 \times c_1 = [a_1, \frac{1}{2}(c_1 - a_1), c_1] = \mathbf{x}$, so property M5 is satisfied. All ten properties are satisfied, so $\mathcal{V} = \{[a, b, c] \mid c - a = 2b,\ a, c \in \mathbb{R}\}$ is a vector space.