Exercise 6

Are the three vectors $[1, 2, 3]$, $[-2, 0, 1]$, and $[1, 10, 17]$ linearly dependent or independent? Do they span all vectors or not?

Solution

By definition, a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be linearly independent when the only solution to the homogeneous equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Otherwise, the set is linearly dependent.

For this particular exercise, $\mathbf{v}_1 = [1, 2, 3]$ and $\mathbf{v}_2 = [-2, 0, 1]$ and $\mathbf{v}_3 = [1, 10, 17]$. There is a nontrivial solution:

$$5\mathbf{v}_1 + 2\mathbf{v}_2 + (-1)\mathbf{v}_3 = \mathbf{0}$$

Therefore, the three vectors are linearly dependent. A more systematic way of coming to this conclusion is the following: Arrange the vectors as the columns (or rows) of a matrix and find the determinant of this matrix. If the determinant is nonzero, then the vectors are linearly independent; otherwise, they are linearly dependent.

$$\begin{aligned}
\det\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 1 & 10 & 17 \end{bmatrix} &= 1\begin{vmatrix} 0 & 1 \\ 10 & 17 \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ 1 & 17 \end{vmatrix} + 3\begin{vmatrix} -2 & 0 \\ 1 & 10 \end{vmatrix} \\
&= 1(-10) - 2(-34 - 1) + 3(-20) \\
&= 0
\end{aligned}$$

This confirms the result. Because $\mathbf{v}_3$ can be written in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$, the vectors only span a plane in $\mathbb{R}^3$, not all of $\mathbb{R}^3$.