Exercise 8

Find a vector that, together with the vectors $[1, 1, 1]$ and $[1, 2, 1]$, forms a basis of $\mathbb{R}^3$.

Solution

By definition, a basis for $\mathbb{R}^3$ is a set of linearly independent vectors that span $\mathbb{R}^3$. In other words, if every vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors in a set, then this set is a basis of $\mathbb{R}^3$.

The vectors $\mathbf{v}_1 = [1, 1, 1]$ and $\mathbf{v}_2 = [1, 2, 1]$ point in two distinct directions in $\mathbb{R}^3$, so it is necessary for the third vector to point in a different direction. Thus, let the third vector be formed from the cross product of the two, $\mathbf{v_3} = \mathbf{v}_1 \times \mathbf{v}_2$. $\mathbf{v}_3$ will point in a direction perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\begin{aligned} \mathbf{v}_3 &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \\ &= \hat{\mathbf{x}} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \hat{\mathbf{y}} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ &= \hat{\mathbf{x}}(-1) + \hat{\mathbf{y}}(0) + \hat{\mathbf{z}}(1) \end{aligned}$$

Therefore, the three vectors, $[1, 1, 1]$, $[1, 2, 1]$, and $[-1, 0, 1]$, form a basis of $\mathbb{R}^3$.