# Exercise 9

Show that the functions $(c_1 + c_2 \sin^2 x + c_3 \cos^2 x)$ form a vector space. Find a basis of it. What is its dimension?

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**Solution**

This is a linear combination of the functions 1, $\sin^2 x$, and $\cos^2 x$. Since $\sin^2 x + \cos^2 x = 1$, the basis is just $\{\sin^2 x, \cos^2 x\}$. And because the basis has two components to it, the dimension is 2. In order to show that the functions form a vector space, we must show that the following ten properties are satisfied:

(A1) $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A2) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{V}$.

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathbb{R}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for every $\alpha \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for all $\alpha, \beta \in \mathbb{R}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

Suppose $\mathbf{x} \in \mathcal{V}$ and $\mathbf{y} \in \mathcal{V}$. Then

$$\begin{aligned}
\mathbf{x} &= a_1 + b_1 \sin^2 x + c_1 \cos^2 x \\
\mathbf{y} &= a_2 + b_2 \sin^2 x + c_2 \cos^2 x
\end{aligned}$$

The sum is $\mathbf{x} + \mathbf{y}$.

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x \\
&= (a_1 + a_2) + (b_1 + b_2) \sin^2 x + (c_1 + c_2) \cos^2 x \\
&= a + b \sin^2 x + c \cos^2 x
\end{aligned}$$

Because $a, b, c \in \mathbb{R}$, $\mathbf{x} + \mathbf{y}$ is just another vector in $\mathcal{V}$; that is, $\mathbf{x} + \mathbf{y} \in \mathcal{V}$, and property A1 is satisfied. Suppose $\mathbf{z} \in \mathcal{V}$. Then $\mathbf{z} = a_3 + b_3 \sin^2 x + c_3 \cos^2 x$, and

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= (a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x) + a_3 + b_3 \sin^2 x + c_3 \cos^2 x \\
&= a_1 + b_1 \sin^2 x + c_1 \cos^2 x + a_2 + b_2 \sin^2 x + c_2 \cos^2 x + a_3 + b_3 \sin^2 x + c_3 \cos^2 x \\
&= a_1 + b_1 \sin^2 x + c_1 \cos^2 x + (a_2 + b_2 \sin^2 x + c_2 \cos^2 x + a_3 + b_3 \sin^2 x + c_3 \cos^2 x) \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z})
\end{aligned}$$

So property A2 is satisfied.

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= a_1 + b_1\sin^2 x + c_1\cos^2 x + a_2 + b_2\sin^2 x + c_2\cos^2 x\\ &= a_2 + b_2\sin^2 x + c_2\cos^2 x + a_1 + b_1\sin^2 x + c_1\cos^2 x\\ &= \mathbf{y}+\mathbf{x}\end{aligned}$$

So property A3 is satisfied. We can choose $a_1 = b_1 = c_1 = 0$ to get the zero vector, so property A4 is satisfied. We can choose $a_2 = -a_1$, $b_2 = -b_1$, and $c_2 = -c_1$ in $\mathbf{y}$ to get $-\mathbf{x}$.

$$\begin{aligned}\mathbf{x}+(-\mathbf{x}) &= a_1 + b_1\sin^2 x + c_1\cos^2 x + (-a_1 - b_1\sin^2 x - c_1\cos^2 x)\\ &= \cancel{a_1} + \cancel{b_1\sin^2 x} + \cancel{c_1\cos^2 x} - \cancel{a_1} - \cancel{b_1\sin^2 x} - \cancel{c_1\cos^2 x}\\ &= 0\end{aligned}$$

So property A5 is satisfied. Choose $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha\mathbf{x} &= \alpha(a_1 + b_1\sin^2 x + c_1\cos^2 x)\\ &= \alpha a_1 + \alpha b_1\sin^2 x + \alpha c_1\cos^2 x\\ &= a + b\sin^2 x + c\cos^2 x\end{aligned}$$

Thus, $\alpha\mathbf{x}$ is just another vector in $\mathcal{V}$. So property M1 is satisfied. Choose $\beta \in \mathbb{R}$. Then

$$\begin{aligned}(\alpha\beta)\mathbf{x} &= (\alpha\beta)(a_1 + b_1\sin^2 x + c_1\cos^2 x)\\ &= \alpha\beta a_1 + \alpha\beta b_1\sin^2 x + \alpha\beta c_1\cos^2 x\\ &= \alpha(\beta a_1 + \beta b_1\sin^2 x + \beta c_1\cos^2 x)\\ &= \alpha(\beta\mathbf{x})\end{aligned}$$

So property M2 is satisfied.

$$\begin{aligned}\alpha(\mathbf{x}+\mathbf{y}) &= \alpha(a_1 + b_1\sin^2 x + c_1\cos^2 x + a_2 + b_2\sin^2 x + c_2\cos^2 x)\\ &= \alpha a_1 + \alpha b_1\sin^2 x + \alpha c_1\cos^2 x + \alpha a_2 + \alpha b_2\sin^2 x + \alpha c_2\cos^2 x\\ &= \alpha(a_1 + b_1\sin^2 x + c_1\cos^2 x) + \alpha(a_2 + b_2\sin^2 x + c_2\cos^2 x)\\ &= \alpha\mathbf{x} + \alpha\mathbf{y}\end{aligned}$$

So property M3 is satisfied.

$$\begin{aligned}(\alpha+\beta)\mathbf{x} &= (\alpha+\beta)(a_1 + b_1\sin^2 x + c_1\cos^2 x)\\ &= (\alpha+\beta)a_1 + (\alpha+\beta)b_1\sin^2 x + (\alpha+\beta)c_1\cos^2 x\\ &= \alpha a_1 + \beta a_1 + \alpha b_1\sin^2 x + \beta b_1\sin^2 x + \alpha c_1\cos^2 x + \beta c_1\cos^2 x\\ &= \alpha a_1 + \alpha b_1\sin^2 x + \alpha c_1\cos^2 x + \beta a_1 + \beta b_1\sin^2 x + \beta c_1\cos^2 x\\ &= \alpha(a_1 + b_1\sin^2 x + c_1\cos^2 x) + \beta(a_2 + b_2\sin^2 x + c_2\cos^2 x)\\ &= \alpha\mathbf{x} + \beta\mathbf{y}\end{aligned}$$

So property M4 is satisfied.

$$\begin{aligned}1\mathbf{x} &= 1(a_1 + b_1\sin^2 x + c_1\cos^2 x)\\ &= 1 * a_1 + 1 * b_1\sin^2 x + 1 * c_1\cos^2 x\\ &= a_1 + b_1\sin^2 x + c_1\cos^2 x\\ &= \mathbf{x}\end{aligned}$$

So property M5 is satisfied. All ten properties are satisfied, and so the functions $(c_1 + c_2\sin^2 x + c_3\cos^2 x)$ form a vector space.