Exercise 10

Show that the solutions of the differential equation $u''' - 3u'' + 4u = 0$ form a vector space. Find a basis of it.

Solution

In order for the solutions of a differential equation to form a vector space $\mathcal{U}$, the vector addition and scalar multiplication operations must satisfy the following ten properties:

(A1) $u_1 + u_2 \in \mathcal{U}$ for all $u_1, u_2 \in \mathcal{U}$.

(A2) $(u_1 + u_2) + u_3 = u_1 + (u_2 + u_3)$ for every $u_1, u_2, u_3 \in \mathcal{U}$.

(A3) $u_1 + u_2 = u_2 + u_1$ for every $u_1, u_2 \in \mathcal{U}$.

(A4) There is an element $0 \in \mathcal{U}$ such that $u_1 + 0 = u_1$ for every $u_1 \in \mathcal{U}$.

(A5) For each $u_1 \in \mathcal{U}$, there is an element $(-u_1) \in \mathcal{U}$ such that $u_1 + (-u_1) = 0$.

(M1) $\alpha u_1 \in \mathcal{U}$ for all $\alpha \in \mathbb{R}$ and $u_1 \in \mathcal{U}$.

(M2) $(\alpha\beta)u_1 = \alpha(\beta u_1)$ for all $\alpha, \beta \in \mathbb{R}$ and every $u_1 \in \mathcal{U}$.

(M3) $\alpha(u_1 + u_2) = \alpha u_1 + \alpha u_2$ for every $\alpha \in \mathbb{R}$ and all $u_1, u_2 \in \mathcal{U}$.

(M4) $(\alpha + \beta)u_1 = \alpha u_1 + \beta u_1$ for all $\alpha, \beta \in \mathbb{R}$ and every $u_1 \in \mathcal{U}$.

(M5) $1u_1 = u_1$ for every $u_1 \in \mathcal{U}$.

Choose $u_1 \in \mathcal{U}$ and $u_2 \in \mathcal{U}$ and $u_3 \in \mathcal{U}$. Then

$$\begin{aligned} u_1''' - 3u_1'' + 4u_1 &= 0 \\ u_2''' - 3u_2'' + 4u_2 &= 0 \\ u_3''' - 3u_3'' + 4u_3 &= 0 \end{aligned}$$

Adding the first two equations together, we get

$$\begin{aligned} u_1''' - 3u_1'' + 4u_1 + u_2''' - 3u_2'' + 4u_2 &= 0 + 0 \\ u_1''' + u_2''' - 3u_1'' - 3u_2'' + 4u_1 + 4u_2 &= 0 \\ (u_1 + u_2)''' - 3(u_1 + u_2)'' + 4(u_1 + u_2) &= 0 \end{aligned}$$

So $u_1 + u_2$ also satisfies the differential equation; in other words, $u_1 + u_2 \in \mathcal{U}$, and property A1 is satisfied.

$$\begin{aligned} [(u_1 + u_2) + u_3]''' - 3[(u_1 + u_2) + u_3]'' + 4[(u_1 + u_2) + u_3] &= 0 \\ [u_1 + u_2 + u_3]''' - 3[u_1 + u_2 + u_3]'' + 4[u_1 + u_2 + u_3] &= 0 \\ [u_1 + (u_2 + u_3)]''' - 3[u_1 + (u_2 + u_3)]'' + 4[u_1 + (u_2 + u_3)] &= 0 \end{aligned}$$

Therefore, property A2 is satisfied.

$$\begin{aligned} (u_1 + u_2)''' - 3(u_1 + u_2)'' + 4(u_1 + u_2) &= 0 \\ (u_2 + u_1)''' - 3(u_2 + u_1)'' + 4(u_2 + u_1) &= 0 \end{aligned}$$

Therefore, property A3 is satisfied. Setting $u = 0$ in the differential equation gives equality, so $0 \in \mathcal{U}$. Property A4 is satisfied.

$$\begin{aligned} u_1''' - 3u_1'' + 4u_1 &= 0 \\ -u_1''' + 3u_1'' - 4u_1 &= 0 \\ (-u_1)''' - 3(-u_1)'' + 4(-u_1) &= 0 \end{aligned}$$

Thus, $-u_1$ is a solution of the differential equation, i.e. $-u_1 \in \mathcal{U}$, and property A5 is satisfied.

Choose $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

$$\begin{aligned} \alpha(u_1''' - 3u_1'' + 4u_1) &= \alpha \times 0 \\ \alpha u_1''' - 3\alpha u_1'' + 4\alpha u_1 &= 0 \\ (\alpha u_1)''' - 3(\alpha u_1)'' + 4(\alpha u_1) &= 0 \end{aligned}$$

So αu_1 also satisfies the differential equation; in other words, $\alpha u_1 \in \mathcal{U}$, and property M1 is satisfied.

$$\begin{aligned} (\alpha\beta)(u_1''' - 3u_1'' + 4u_1) &= (\alpha\beta) \times 0 \\ (\alpha\beta)u_1''' - 3(\alpha\beta)u_1'' + 4(\alpha\beta)u_1 &= 0 \\ [(\alpha\beta)u_1]''' - 3[(\alpha\beta)u_1]'' + 4[(\alpha\beta)u_1] &= 0 \\ [\alpha(\beta u_1)]''' - 3[\alpha(\beta u_1)]'' + 4[\alpha(\beta u_1)] &= 0 \end{aligned}$$

Both $(\alpha\beta)u_1$ and $\alpha(\beta u_1)$ satisfy the differential equation, so $(\alpha\beta)u_1 = \alpha(\beta u_1)$. And property M2 is satisfied.

$$\begin{aligned} [\alpha(u_1 + u_2)]''' - 3[\alpha(u_1 + u_2)]'' + 4[\alpha(u_1 + u_2)] &= 0 \\ \alpha(u_1''' - 3u_1'' + 4u_1 + u_2''' - 3u_2'' + 4u_2) &= 0 \\ \alpha u_1''' - 3\alpha u_1'' + 4\alpha u_1 + \alpha u_2''' - 3\alpha u_2'' + 4\alpha u_2 &= 0 \\ \alpha(u_1''' - 3u_1'' + 4u_1) + \alpha(u_2''' - 3u_2'' + 4u_2) &= 0 \end{aligned}$$

Thus, property M3 is satisfied.

$$\begin{aligned} (\alpha + \beta)(u_1''' - 3u_1'' + 4u_1) &= (\alpha + \beta) \times 0 \\ (\alpha + \beta)u_1''' - 3(\alpha + \beta)u_1'' + 4(\alpha + \beta)u_1 &= 0 \\ \alpha u_1''' - 3\alpha u_1'' + 4\alpha u_1 + \beta u_1''' - 3\beta u_1'' + 4\beta u_1 &= 0 \\ \alpha(u_1''' - 3u_1'' + 4u_1) + \beta(u_1''' - 3u_1'' + 4u_1) &= 0 \end{aligned}$$

And so property M4 holds.

$$\begin{aligned} (1u_1)''' - 3(1u_1)'' + 4(1u_1) &= 0 \\ 1 * u_1''' - 1 * 3u_1'' + 1 * 4u_1 &= 0 \\ u_1''' - 3u_1'' + 4u_1 &= 0 \end{aligned}$$

And so property M5 is satisfied. All ten properties are satisfied, so the solutions of the differential equation form a vector space.

The equation $u''' - 3u'' + 4u = 0$ is a linear ordinary differential equation with constant coefficients. Therefore, the solutions are of the form $u(x) = e^{rx}$. Plugging this into the equation,

$$\begin{aligned} r^3e^{rx} - 3r^2e^{rx} + 4e^{rx} &= 0 \\ r^3 - 3r^2 + 4 &= 0 \\ (r+1)(r^2 - 4r + 4) &= 0 \\ (r+1)(r-2)^2 &= 0 \\ \rightarrow \quad r &= \{-1, 2\} \end{aligned}$$

So the solution to the differential equation is

$$u(x) = c_1e^{-x} + c_2e^{2x} + c_3xe^{2x}$$

The basis of the vector space formed by the solutions is therefore $\{e^{-x}, e^{2x}, xe^{2x}\}$.