Exercise 2

Solve the equation $3u_y + u_{xy} = 0$. (*Hint*: Let $v = u_y$.)

Solution

Following the given hint, let $v = u_y$. The PDE transforms to

$$3v + v_x = 0$$

This is a first-order linear differential equation that can be solved with an integrating factor.

$$I = e^{\int 3\,dx} = e^{3x}$$

Multiplying both sides of the equation by this integrating factor gives

$$e^{3x}v_x + 3e^{3x}v = 0.$$

But the left side is simply the x-derivative of the product, $e^{3x}v$.

$$\partial_x(e^{3x}v) = 0$$

Integrating both sides partially with respect to x gives

$$e^{3x}v = f(y),$$

where f is an arbitrary function of y. Solving for v gives

$$\begin{aligned} v(x,y) &= e^{-3x}f(y) \\ u_y(x,y) &= e^{-3x}f(y). \end{aligned}$$

To solve for u, integrate both sides partially with respect to y.

$$\begin{aligned} u(x,y) &= \int^y e^{-3x}f(s)\,ds + g(x) \\ &= e^{-3x}\int^y f(s)\,ds + g(x) \end{aligned}$$

Therefore,

$$u(x,y) = e^{-3x}F(y) + g(x),$$

where F is an arbitrary function of y and g is an arbitrary function of x. We can check to see if this solves the PDE.

$$\begin{aligned} u_y &= e^{-3x}F'(y) \\ u_{yx} = u_{xy} &= -3e^{-3x}F'(y) \end{aligned}$$

And so, $3u_y + u_{xy} = 0$. This is indeed the correct solution.