Exercise 1

Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

Solution

The Geometric Method: Characteristic Curves

Start by rewriting the PDE as

$$u_t + \frac{3}{2}u_x = 0$$

and then apply the method of characteristics to solve for u. On the paths defined by

$$\frac{dx}{dt} = \frac{3}{2}, \quad x(\xi, 0) = \xi, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dt} = 0. \tag{2}$$

That is, $u = u(x, t)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(\xi, t) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Integrating (1), we see that

$$x = \frac{3}{2}t + \xi.$$

Solving for ξ gives

$$\xi = x - \frac{3}{2}t.$$

Therefore,

$$u(x, t) = f\left(x - \frac{3}{2}t\right).$$

We can check that this is the solution to the PDE.

$$u_x = f'$$
$$u_t = -\frac{3}{2}f'$$

$2u_t + 3u_x = 0$, so this is the solution to the PDE. We're told that $u(x, 0) = \sin x$, though, so we can determine this unknown function, f.

$$u(x, 0) = f(x) = \sin x$$

This implies that $f(w) = \sin w$, where w is any expression. Thus,

$$u(x, t) = \sin\left(x - \frac{3}{2}t\right).$$

The function is shown below in Figure 1. Shown below that in Figure 2 are the characteristic curves in the tx-plane for various values of ξ along with the line $t = 0$ (where the auxiliary condition is defined). Note that because $t = 0$ intersects each of the characteristics exactly once, the solution we obtained for $u(x, t)$ is valid everywhere in the tx-plane, that is, for all x and t.

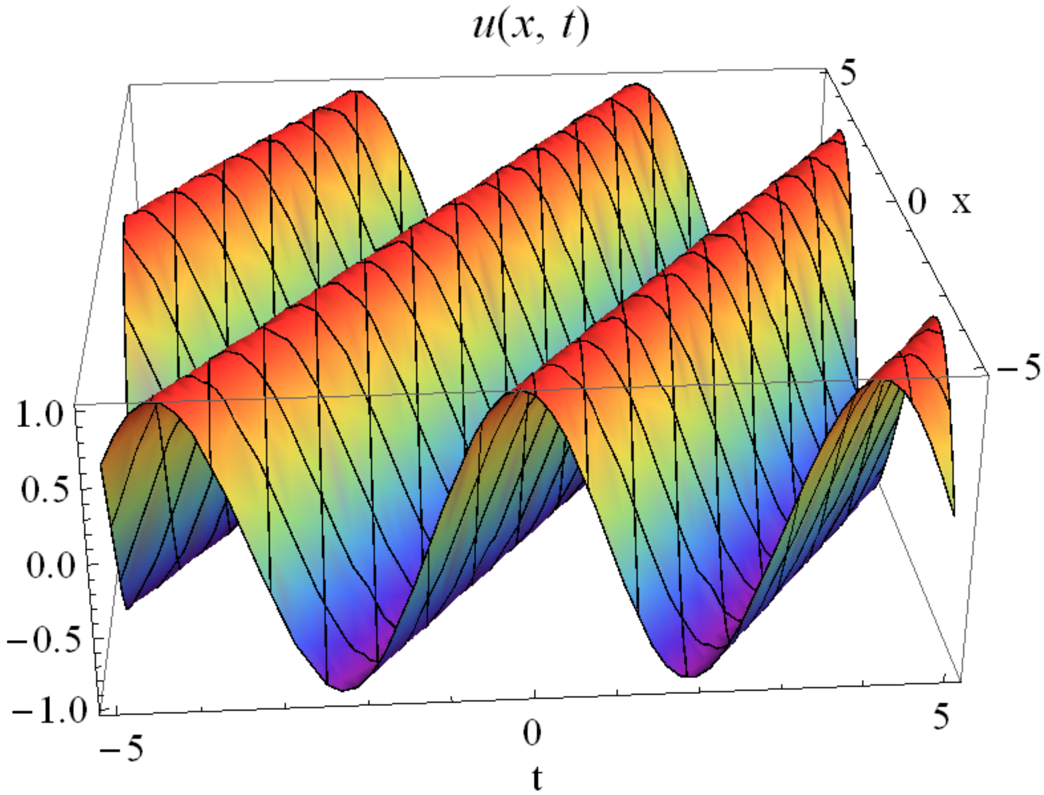

Figure 1: Plot of $u(x,t)$ for $-5 < t < 5$ and $-5 < x < 5$.

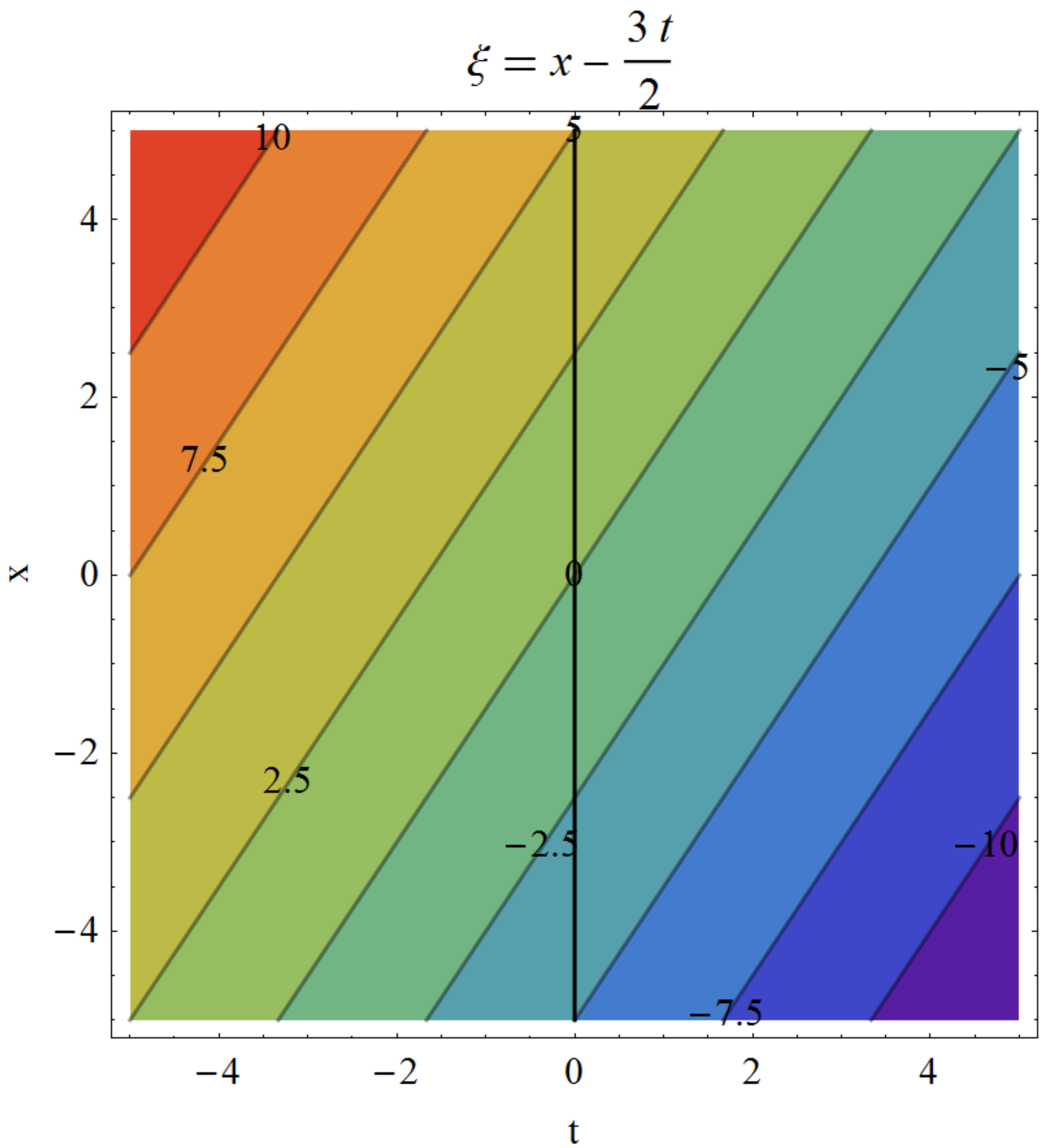

Figure 2: Plot of the characteristic curves along with the data curve in the tx-plane.

The Coordinate Method: Change of Variables

To solve this PDE with the coordinate method, start by making the change of variables,

$$t' = 2t + 3x$$
$$x' = 3t - 2x.$$

Solving for the old variables in terms of the new ones gives us

$$t = \frac{1}{13}(2t' + 3x')$$
$$x = \frac{1}{13}(3t' - 2x').$$

To find what u_t and u_x are in terms of these new variables, it's necessary to use the chain rule.

$$u_t = \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial t} = 2u_{t'} + 3u_{x'}$$
$$u_x = \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial x} + \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} = 3u_{t'} - 2u_{x'}$$

Now we substitute these expressions into the PDE. The transformed equation is

$$2(2u_{t'} + 3u_{x'}) + 3(3u_{t'} - 2u_{x'}) = 0.$$

Simplifying this gives

$$13u_{t'} = 0$$
$$u_{t'} = 0.$$

Solve for u by partially integrating both sides with respect to t'.

$$u(x', t') = g(x'),$$

where g is an arbitrary function of x'. Now we return to the original variables, x and t.

$$u(x,t) = g(3t - 2x)$$

We can check that this is the solution.

$$u_t = 3g'$$
$$u_x = -2g'$$

$2u_t + 3u_x = 0$, which means this is the solution to the PDE. Now plug in the initial condition, $u(x, 0) = \sin x$ to determine g.

$$u(x, 0) = g(-2x) = \sin x$$

This implies that

$$g(w) = \sin\left(-\frac{w}{2}\right),$$

where w is any expression. Therefore,

$$u(x,t) = \sin\left(-\frac{3t - 2x}{2}\right)$$

$$u(x,t) = \sin\left(x - \frac{3}{2}t\right),$$

and we get the same answer as with the method of characteristics.