Exercise 3

Solve the equation $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.

Solution

Start by rewriting the PDE as

$$u_x + \frac{1}{1+x^2}u_y = 0$$

and then apply the method of characteristics to solve for u. On the paths defined by

$$\frac{dy}{dx} = \frac{1}{1+x^2}, \quad y(\xi, 0) = \xi, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(x, \xi) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Integrating (1) gives

$$y = \tan^{-1} x + \xi.$$

Solving for ξ gives

$$\xi = y - \tan^{-1} x.$$

Therefore,

$$u(x, y) = f\left(y - \tan^{-1} x\right).$$

We can check that this is the solution of the PDE.

$$u_x = -\frac{1}{1+x^2}f'$$
$$u_y = f'$$

$(1 + x^2)u_x + u_y = 0$, so this is the correct solution. Shown in Figure 1 are the characteristic curves in the xy-plane for various values of ξ.

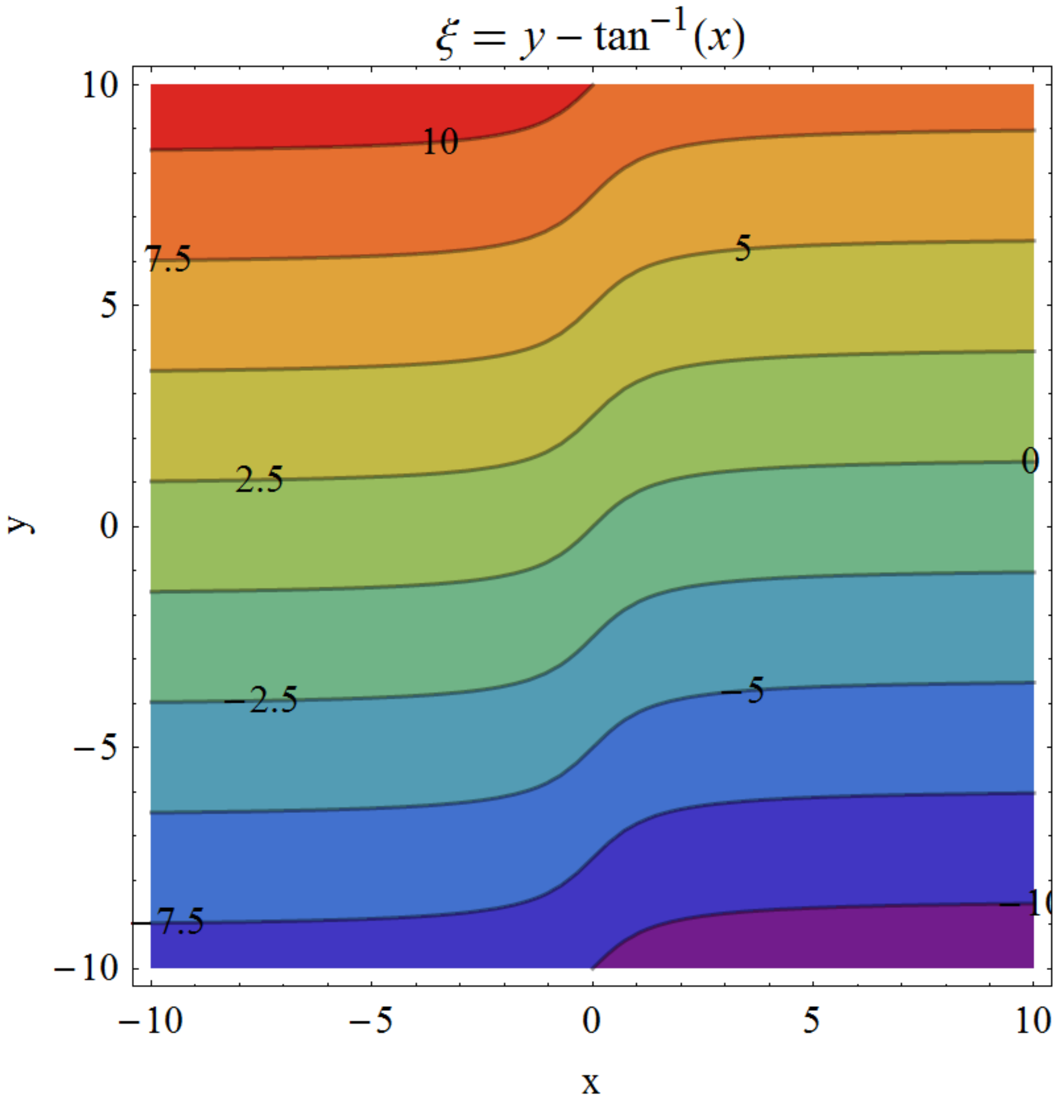

Figure 1: Plot of the characteristic curves in the xy-plane.