Exercise 5

Solve the equation $xu_x + yu_y = 0$.

Solution

Start by rewriting the PDE as

$$u_x + \frac{y}{x}u_y = 0$$

and then apply the method of characteristics to solve for u. On the paths defined by

$$\frac{dy}{dx} = \frac{y}{x}, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(x, \xi) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Solving (1) by separation of variables gives

$$\begin{aligned}\frac{dy}{y} &= \frac{dx}{x} \\ \ln|y| &= \ln|x| + \xi\end{aligned}$$

Solving for ξ gives

$$\xi = \ln|y| - \ln|x| = \ln\left|\frac{y}{x}\right|.$$

So

$$u(x, y) = f\left(\ln\left|\frac{y}{x}\right|\right).$$

But this can be simplified. We can say that $u(x, y)$ is really just some function of y/x.

$$u(x, y) = g\left(\frac{y}{x}\right),$$

where g is an arbitrary function. We can check that this is the solution of the PDE.

$$\begin{aligned}u_x &= -\frac{y}{x^2}g' \\ u_y &= \frac{1}{x}g'\end{aligned}$$

$xu_x + yu_y = 0$, so this is the correct solution. Shown in Figure 1 are the characteristic curves in the xy-plane for various values of ξ.

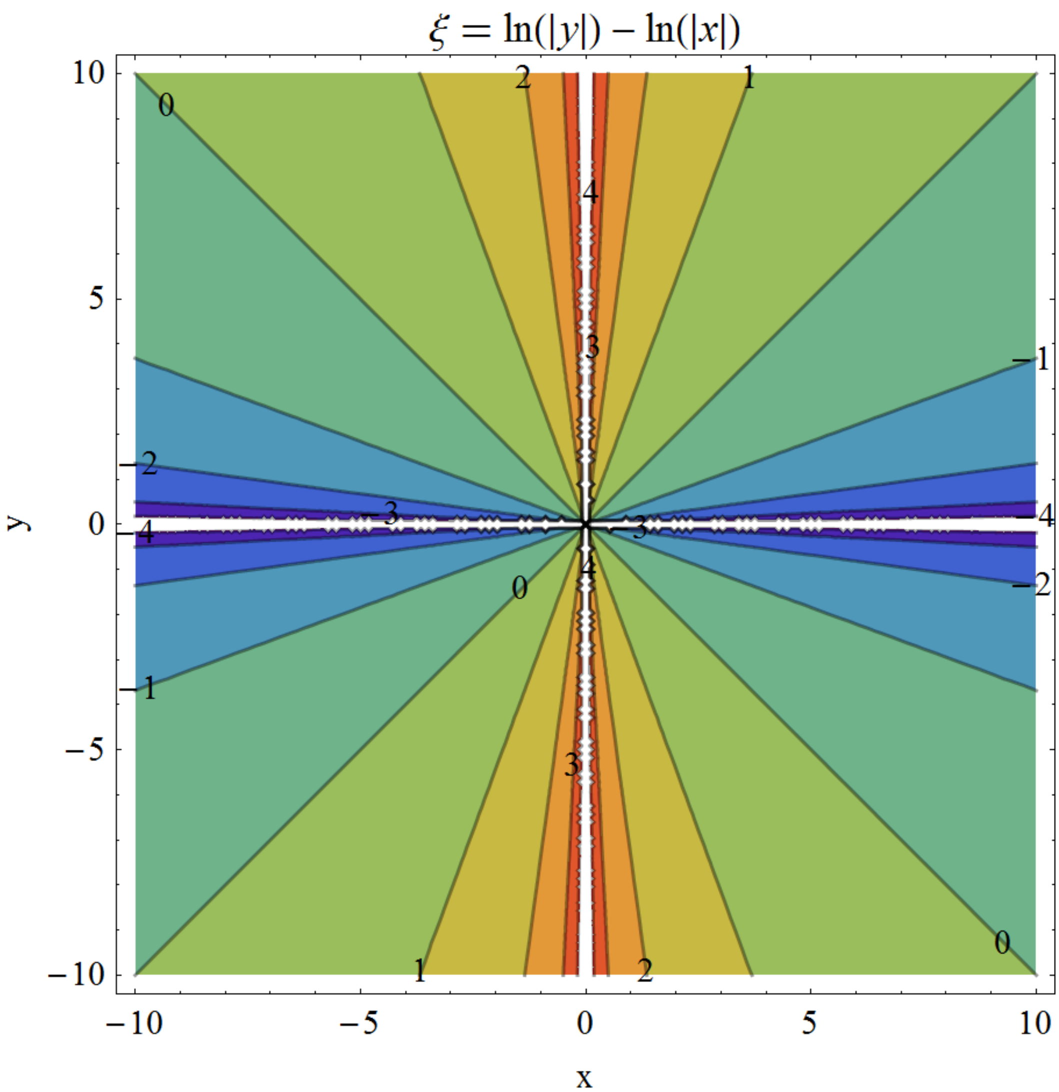

Figure 1: Plot of the characteristic curves in the xy-plane.