Exercise 7

(a) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.

(b) In which region of the xy plane is the solution uniquely determined?

Solution

Part (a)

Start by rewriting the PDE as

$$u_x + \frac{x}{y}u_y = 0$$

and then apply the method of characteristics to solve for u. On the paths defined by

$$\frac{dy}{dx} = \frac{x}{y}, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(x, \xi) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Solving (1) with separation of variables gives

$$y\,dy = x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C.$$

Multiply both sides by 2 to get rid of the fractions, and let ξ be the constant of integration.

$$y^2 = x^2 + \xi$$

Solving for ξ gives

$$\xi = y^2 - x^2.$$

Therefore,

$$u(x, y) = f\left(y^2 - x^2\right).$$

We're told that $u(0, y) = e^{-y^2}$, though, so we can determine this unknown function, f.

$$u(0, y) = f(y^2) = e^{-y^2}$$

This implies that $f(w) = e^{-w}$, where w is any expression. Thus,

$$u(x, y) = e^{-(y^2 - x^2)}.$$

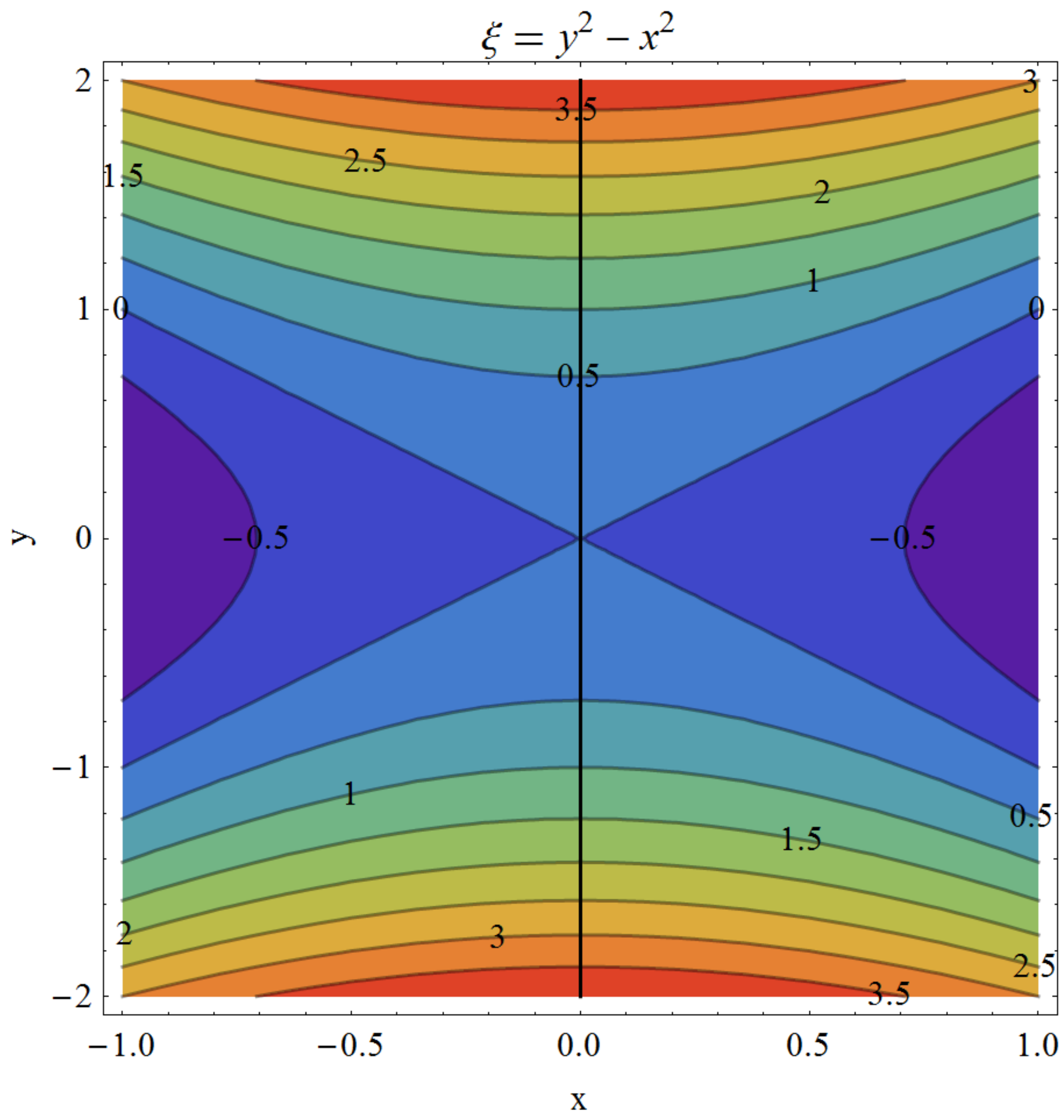

Figure 1: Plot of the characteristic curves along with the data curve in the xy-plane.

Part (b)

The data curve $x = 0$ only intersects the characteristics with a nonnegative ξ value exactly once. All the characteristics with a negative value for ξ, that is, those in the dark blue and purple regions never touch the data curve. Therefore, the solution we obtained in the previous part is only valid for $\xi = y^2 - x^2 \geq 0$. In the part of the xy-plane where $y^2 - x^2 < 0$, the function f is undetermined.

$$u(x, y) = \begin{cases} e^{-(y^2 - x^2)} & \text{for } y^2 - x^2 \geq 0 \\ f\left(y^2 - x^2\right) & \text{for } y^2 - x^2 < 0 \end{cases}$$

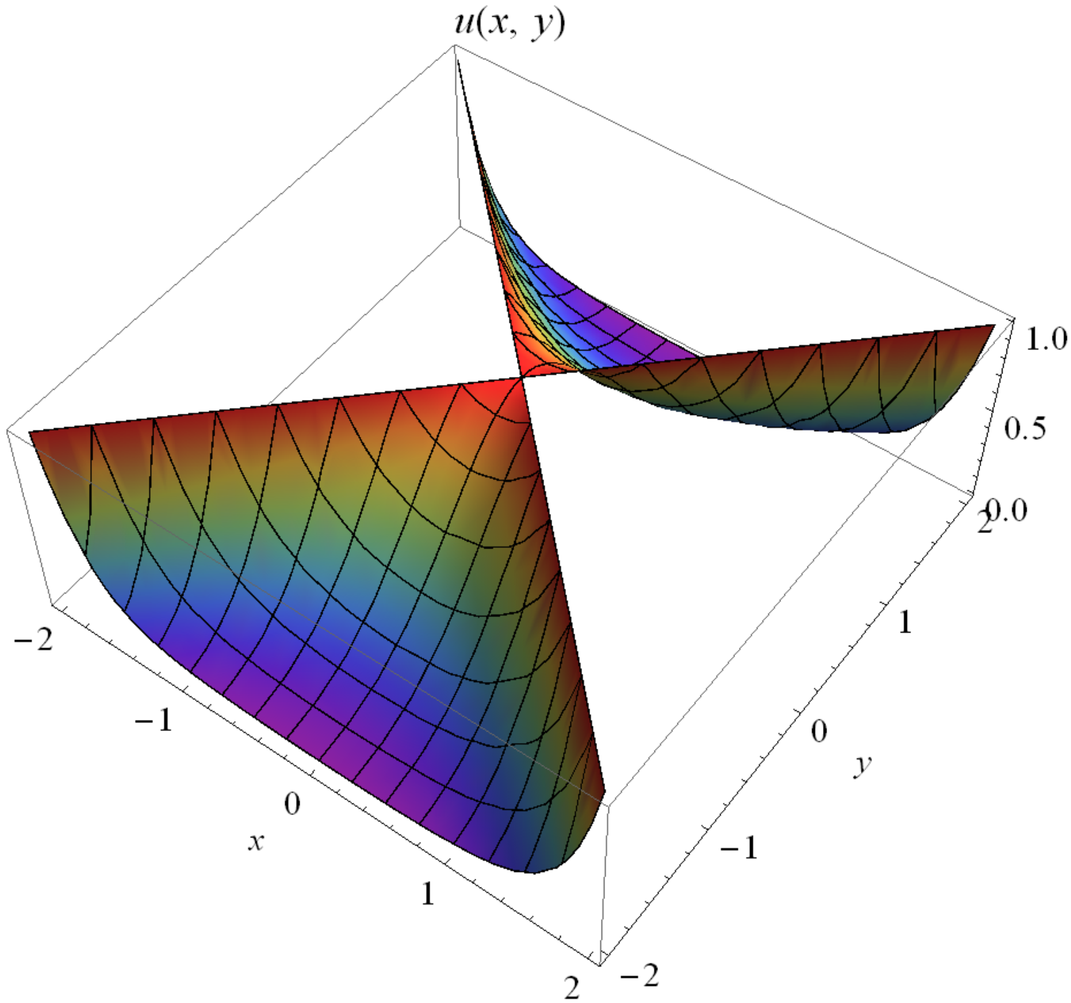

Figure 2: Plot of $u(x,y)$ for $-2 < x < 2$ and $-2 < y < 2$. Note that this graph is defined only for $y^2 - x^2 \geq 0$.

We can check that this is the solution of the PDE.

$$u_x = -2xf'$$
$$u_y = 2yf'$$

$yu_x + xu_y = 0$, so this is the correct solution.