Exercise 9

Solve the equation $u_x + u_y = 1$.

Solution

The Geometric Method: Characteristic Curves

On the paths defined by

$$\frac{dy}{dx} = 1, \quad y(\xi, 0) = \xi, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} = 1. \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(x, \xi) = x + f(\xi).$$

where f is an arbitrary function of the characteristic coordinate, ξ. Integrating (1) gives

$$y = x + \xi$$

Solving for ξ gives

$$\xi = y - x.$$

Therefore,

$$u(x, y) = x + f(y - x).$$

We can check that this is the solution.

$$u_x = 1 - f'$$
$$u_y = f'$$

$u_x + u_y = 1$, which means this is the solution to the PDE.

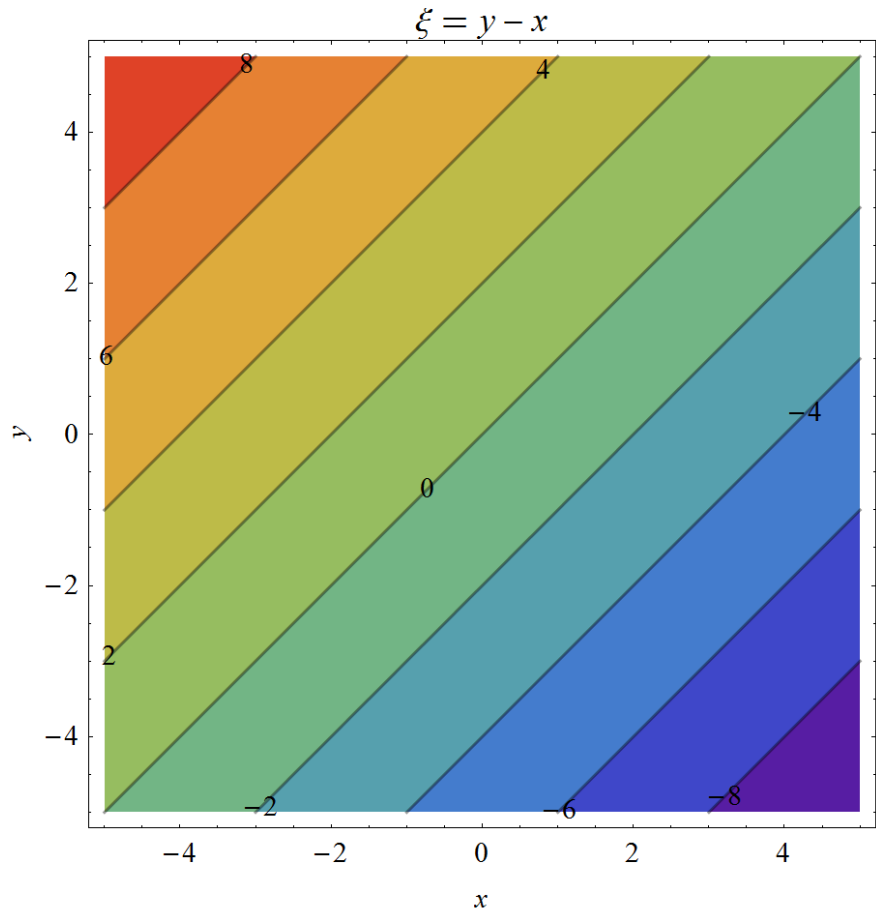

Figure 1: Plot of the characteristic curves in the xy-plane for $-5 < x < 5$ and $-5 < y < 5$.

The Coordinate Method: Change of Variables

To solve this PDE with the coordinate method, start by making the change of variables,

$$x' = x + y$$
$$y' = x - y.$$

Solving for the old variables in terms of the new ones gives us

$$x = \frac{1}{2}(x' + y')$$
$$y = \frac{1}{2}(x' - y').$$

To find what u_x and u_y are in terms of these new variables, it's necessary to use the chain rule.

$$u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = u_{x'} + u_{y'}$$
$$u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = u_{x'} - u_{y'}$$

Now we substitute these expressions into the PDE. The transformed equation is

$$(u_{x'} + u_{y'}) + (u_{x'} - u_{y'}) = 1.$$

Simplifying this gives

$$\begin{aligned} 2u_{x'} &= 1 \\ u_{x'} &= \frac{1}{2}. \end{aligned}$$

Solve for u by partially integrating both sides with respect to x'.

$$u(x', y') = \frac{1}{2}x' + g(y'),$$

where g is an arbitrary function of y'. Now we return to the original variables, x and y.

$$u(x, y) = \frac{1}{2}(x + y) + g(x - y)$$

We can check that this is the solution.

$$\begin{aligned} u_x &= \frac{1}{2} + g' \\ u_y &= \frac{1}{2} - g' \end{aligned}$$

$u_x + u_y = 1$, which means this is the solution to the PDE.