Exercise 1

Carefully derive the equation of a string in a medium in which the resistance is proportional to the velocity.

Solution

There are two ways (among others) to go about this problem. One is the integral formulation, where we consider the forces acting over a finite portion of the string. The other is the differential formulation, where we consider the forces acting on an infinitesimal element of the string. In both cases we come to the same governing PDE, so use whichever you prefer.

The Integral Formulation

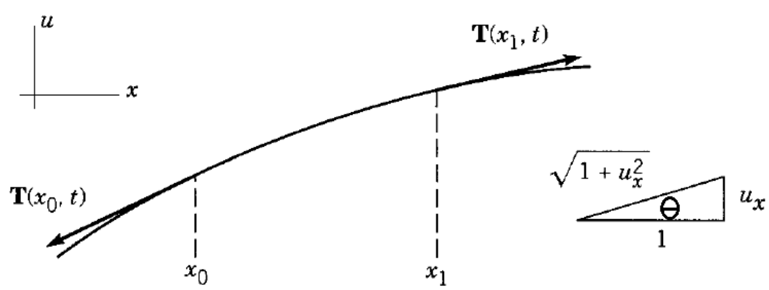

Figure 1: Schematic of the string (integral formulation).

In order to derive the equation of motion, we will invoke Newton's second law, which states that the sum of the forces acting on a body is equal to its mass times its acceleration. Mathematically this is written as

$$\sum \mathbf{F} = m\mathbf{a}.$$

Note that this is a vector equation; in other words, there is a separate equation for each component of force and corresponding component of acceleration. For this problem we will choose the coordinate system as shown in the figure, so there are two equations of significance.

$$\sum F_x = ma_x$$
$$\sum F_u = ma_u$$

There are two forces acting on this string, $\mathbf{T}$ at $x = x_0$ and $\mathbf{T}$ at $x = x_1$, in addition to the resistive force of the medium. The motion of the string is entirely vertical, which means there is no horizontal component of acceleration ($a_x = 0$). Hence, the resistive force of the medium will only act vertically. The tensions, on the other hand, have components in both the x-direction and u-direction and have to be resolved using cosine and sine, respectively.

Horizontal component of $\mathbf{T}$ at $x = x_0$:	$-T\cos\theta_0$
Horizontal component of $\mathbf{T}$ at $x = x_1$:	$+T\cos\theta_1$
Vertical component of $\mathbf{T}$ at $x = x_0$:	$-T\sin\theta_0$
Vertical component of $\mathbf{T}$ at $x = x_1$:	$+T\sin\theta_1$,

where θ_0 and θ_1 are the angles between the vectors and the x-axis at x_0 and x_1, respectively. To determine θ it is necessary to note that $\tan\theta$ is equal to rise over run, the slope. If the height of the string is $u(x,t)$, the slope is given by $\partial u/\partial x = u_x$. As shown in Figure 1, the hypotenuse can be determined using Pythagorean's theorem. And now $\cos\theta$ can be written in terms of u. Newton's second law in the x-direction is thus

$$\sum F_x = -T\cos\theta_0 + T\cos\theta_1 = ma_x = 0$$

$$-T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_0} + T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_1} = 0.$$

Hence,

$$T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_0} = T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_1}.$$

This equation can be simplified if we make the assumption that u, and therefore u_x, remains small for all x and t. The binomial theorem tells us that

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u_x^4 + \cdots.$$

Compared to 1, u_x^2 and all higher powers of u_x can be considered negligible. Approximating the square root terms as 1, the equation of motion simplifies to

$$T|_{x=x_0} \approx T|_{x=x_1}.$$

This equation tells us that the magnitude of the tension at x_0 is equal to the magnitude of the tension at x_1; that is, T is constant. Let's move on to Newton's second law in the u-direction. If the height of the string is $u(x,t)$, the rate of change of the height with respect to time, the velocity, is given by $\partial u/\partial t = u_t$. Consequently, the rate of change of velocity with respect to time, the acceleration, is $\partial^2 u/\partial t^2 = u_{tt}$. As said in the problem statement, the resistive force is proportional to velocity. Mathematically this can be expressed as

$$\underbrace{F_r}_{\text{resistive force}} \quad \underbrace{\propto}_{\text{proportional to}} \quad -\underbrace{u_t}_{\text{velocity}}.$$

To change this proportionality to an equation we can use, we have to introduce a constant of proportionality. Let's call it r. The minus sign indicates that the force acts in the direction that opposes the motion.

$$F_r = -ru_t$$

Understand that this force is applied at a specific point x at time t on the string. To get the total force acting on the string from $x = x_0$ to $x = x_1$, we have to sum (integrate) the resistive forces (F_r) from x_0 to x_1. Think of F_r as a distributed force that we have to integrate over x to get the total of. Note, too, that u_{tt} is the acceleration at a specific point on the string, x, at time t. To get the force we therefore have to multiply u_{tt} by a tiny bit of mass dm. Mass, of course, is density times length, so this can be written in terms of arc length, s, as $dm = \rho\, ds$. The total force is obtained by integrating $u_{tt}\, dm$ over the mass of the string. Newton's second law in the u-direction is thus

$$\sum F_u = -T\sin\theta_0 + T\sin\theta_1 + \int_{x_0}^{x_1} F_r\, dx = \int_{\text{mass of string}} u_{tt}\, dm.$$

Write θ in terms of u using the right triangle in Figure 1 and substitute $dm = \rho\, ds$.

$$-T \left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=x_0} + T \left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=x_1} - \int_{x_0}^{x_1} r u_t\, dx = \int_{\text{length of string}} \rho u_{tt}\, ds = \int_{x_0}^{x_1} \rho u_{tt} \sqrt{1+u_x^2}\, dx$$

This equation can be simplified if we make the assumption that u, and therefore u_x, remains small for all x and t. The binomial theorem tells us that

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u_x^4 + \cdots .$$

Compared to 1, u_x^2 and all higher powers of u_x can be considered negligible. Approximating the square root terms as 1, the equation of motion simplifies to

$$Tu_x(x_1,t) - Tu_x(x_0,t) - \int_{x_0}^{x_1} r u_t\, dx \approx \int_{x_0}^{x_1} \rho u_{tt}\, dx.$$

According to the fundamental theorem of calculus,

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

so the left side can be written as

$$Tu_x(x_1,t) - Tu_x(x_0,t) = \int_{x_0}^{x_1} \frac{\partial}{\partial x}(Tu_x)\, dx.$$

Hence,

$$\int_{x_0}^{x_1} \frac{\partial}{\partial x}(Tu_x)\, dx - \int_{x_0}^{x_1} r u_t\, dx = \int_{x_0}^{x_1} \rho u_{tt}\, dx$$

$$\int_{x_0}^{x_1} \left[\frac{\partial}{\partial x}(Tu_x) - r u_t\right] dx = \int_{x_0}^{x_1} \rho u_{tt}\, dx.$$

Thus, the integrands must be equal to each other.

$$\frac{\partial}{\partial x}(Tu_x) - r u_t = \rho u_{tt}$$

The tension is constant, so

$$Tu_{xx} - r u_t = \rho u_{tt}.$$

Therefore,

$$u_{tt} + 2k u_t = c^2 u_{xx},$$

where $2k = r/\rho$ and $c^2 = T/\rho$.

The Differential Formulation

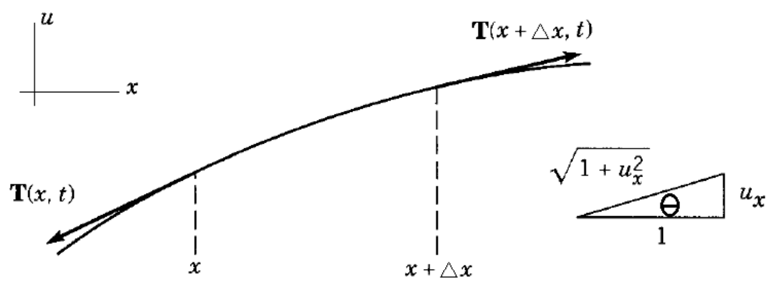

Figure 2: Schematic of the string (differential formulation).

In order to derive the equation of motion, we will invoke Newton's second law, which states that the sum of the forces acting on a body is equal to its mass times its acceleration. Mathematically this is written as

$$\sum \mathbf{F} = m\mathbf{a}.$$

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Horizontal component of $\mathbf{T}$ at x:	$-T\cos\theta_x$
Horizontal component of $\mathbf{T}$ at $x + \Delta x$:	$+T\cos\theta_{x+\Delta x}$
Vertical component of $\mathbf{T}$ at x:	$-T\sin\theta_x$
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where θ_x and $\theta_{x+\Delta x}$ are the angles between the vectors and the x-axis at x and $x + \Delta x$, respectively. To determine θ it is necessary to note that $\tan\theta$ is equal to rise over run, the slope. If the height of the string is $u(x,t)$, the slope is given by $\partial u/\partial x = u_x$. As shown in Figure 2, the hypotenuse can be determined using Pythagorean's theorem. And now $\cos\theta$ can be written in terms of u. Newton's second law in the x-direction is thus

$$\sum F_x = -T\cos\theta_x + T\cos\theta_{x+\Delta x} = ma_x = 0$$
$$-T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_x + T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x+\Delta x} = 0.$$

Hence,

$$T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_x = T\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x+\Delta x}.$$

This equation can be simplified if we make the assumption that u, and therefore u_x, remains small for all x and t. The binomial theorem tells us that

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u_x^4 + \cdots.$$

Compared to 1, u_x^2 and all higher powers of u_x can be considered negligible. Approximating the square root terms as 1, the equation of motion simplifies to

$$T|_x \approx T|_{x+\Delta x}.$$

This equation tells us that the magnitude of the tension at x is equal to the magnitude of the tension at $x + \Delta x$; that is, T is constant. Let's move on to Newton's second law in the u-direction. If the height of the string is $u(x,t)$, the rate of change of the height with respect to time, the velocity, is given by $\partial u/\partial t = u_t$. Consequently, the rate of change of velocity with respect to time, the acceleration, is $\partial^2 u/\partial t^2 = u_{tt}$. As said in the problem statement, the resistive force is proportional to velocity. Mathematically this can be expressed as

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To change this proportionality to an equation we can use, we have to introduce a constant of proportionality. Let's call it r. The minus sign indicates that the force acts in the direction that opposes the motion.

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Understand that this force is applied at a specific point x at time t on the string. To get the total force acting on the string from x to $x + \Delta x$, we have to sum (integrate) the resistive forces (F_r) from x to $x + \Delta x$. Think of F_r as a distributed force that we have to integrate over x to get the total of. Note, too, that u_{tt} is the acceleration at a specific point on the string, x, at time t. To get the force we therefore have to multiply u_{tt} by a tiny bit of mass Δm. Mass, of course, is density times length, so this can be written in terms of arc length, s, as $\Delta m = \rho\,\Delta s$. Newton's second law in the u-direction is thus

$$\sum F_u = -T\sin\theta_x + T\sin\theta_{x+\Delta x} + \int_x^{x+\Delta x} F_r\,dx = u_{tt}\,\Delta m.$$

Write θ in terms of u using the right triangle in Figure 2 and substitute $\Delta m = \rho\,\Delta s$.

$$-T\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_x + T\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x+\Delta x} - \int_x^{x+\Delta x} ru_t\,dx = \rho u_{tt}\,\Delta s = \rho u_{tt}\sqrt{1+u_x^2}\,\Delta x$$

This equation can be simplified if we make the assumption that u, and therefore u_x, remains small for all x and t. The binomial theorem tells us that

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u_x^4 + \cdots.$$

Compared to 1, u_x^2 and all higher powers of u_x can be considered negligible. Approximating the square root terms as 1, the equation of motion becomes

$$Tu_x(x+\Delta x,t) - Tu_x(x,t) - \int_x^{x+\Delta x} ru_t\,dx \approx \rho u_{tt}\,\Delta x.$$

Now take the limit of both sides as $\Delta x \to 0$. The integrand is constant from x to $x+\Delta x$ in the limit, so it can be pulled out in front of the integral. So

$$\lim_{\Delta x\to 0}[Tu_x(x+\Delta x,t) - Tu_x(x,t)] - \lim_{\Delta x\to 0} ru_t \int_x^{x+\Delta x} dx = \lim_{\Delta x\to 0} \rho u_{tt}\,\Delta x$$

$$\lim_{\Delta x\to 0}[Tu_x(x+\Delta x,t) - Tu_x(x,t)] - ru_t \lim_{\Delta x\to 0} \Delta x = \rho u_{tt} \lim_{\Delta x\to 0} \Delta x.$$

Now divide both sides by Δx.

$$\lim_{\Delta x\to 0} \frac{Tu_x(x+\Delta x,t) - Tu_x(x,t)}{\Delta x} - ru_t = \rho u_{tt}$$

According to the definition of the derivative,

$$\frac{df}{dx} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Hence,

$$\frac{\partial}{\partial x}(Tu_x) - ru_t = \rho u_{tt}$$

The tension is constant, so

$$Tu_{xx} - ru_t = \rho u_{tt}.$$

Therefore,

$$u_{tt} + 2ku_t = c^2 u_{xx},$$

where $2k = r/\rho$ and $c^2 = T/\rho$.

P.S. The solution to this PDE is quite involved but very interesting. It can be found in the solution to exercise 12 of section 3.3 in Asmar's "Partial Differential Equations with Fourier Series and Boundary Value Problems."