Exercise 2

A flexible chain of length l is hanging from one end $x = 0$ but oscillates horizontally. Let the x axis point downward and the u axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.

Solution

There are two ways (among others) to go about this problem. One is the integral formulation, where we consider the forces acting over a finite portion of the chain. The other is the differential formulation, where we consider the forces acting on an infinitesimal element of the chain. In both cases we come to the same governing PDE, so use whichever you prefer. This exercise is known as the hanging chain problem. The only difference between this exercise and the previous one is that here the tension is not constant—it will vary as a function of x, which means it will remain inside the x-derivative with u_x at the end.

The Integral Formulation

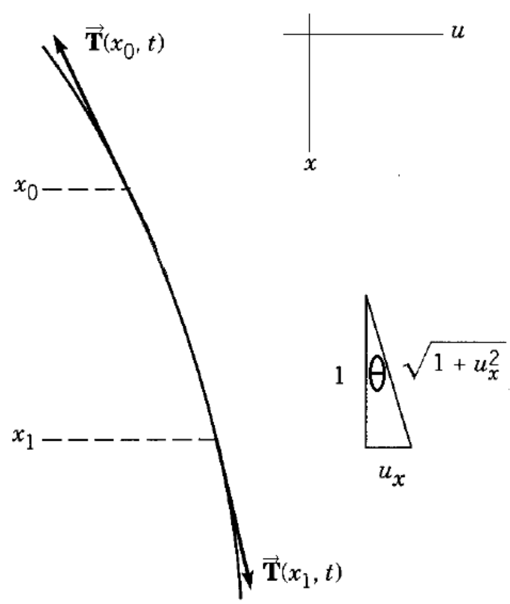

Figure 1: Schematic of the chain (integral formulation).

In order to derive the equation of motion, we will invoke Newton's second law, which states that the sum of the forces acting on a body is equal to its mass times its acceleration. Mathematically this is written as

$$\sum \mathbf{F} = m\mathbf{a}.$$

Note that this is a vector equation; in other words, there is a separate equation for each component of force and corresponding component of acceleration. For this problem we will choose the coordinate system as shown in Figure 1, so there are two equations of significance.

$$\sum F_x = ma_x$$
$$\sum F_u = ma_u$$

There are two forces acting on this chain, $\mathbf{T}$ at $x = x_0$ and $\mathbf{T}$ at $x = x_1$. Because the tension only depends on the weight of the string below it, it is a function of x only. The motion of the chain is entirely in the u-direction, which means there is no vertical component of acceleration ($a_x = 0$). The tensions at x_0 and x_1 have components in both the x-direction and u-direction and have to be resolved using cosine and sine, respectively.

Component of $\mathbf{T}$ in x-direction at $x = x_0$:	$-T(x_0)\cos\theta_0$
Component of $\mathbf{T}$ in x-direction at $x = x_1$:	$+T(x_1)\cos\theta_1$
Component of $\mathbf{T}$ in u-direction at $x = x_0$:	$-T(x_0)\sin\theta_0$
Component of $\mathbf{T}$ in u-direction at $x = x_1$:	$+T(x_1)\sin\theta_1$,

where θ_0 and θ_1 are the respective angles between the vectors and the x-axis. To determine θ it is necessary to note that $\tan\theta$ is equal to rise over run, the slope. If the height of the chain is $u(x,t)$, the slope is given by $\partial u/\partial x = u_x$. As shown in Figure 1, the hypotenuse can be determined using Pythagorean's theorem. And now $\cos\theta$ can be written in terms of u. Newton's second law in the x-direction is thus

$$\sum F_x = -T(x_0)\cos\theta_0 + T(x_1)\cos\theta_1 = ma_x = 0$$
$$-T(x_0)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_0} + T(x_1)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_1} = 0.$$

Hence,

$$T(x_0)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_0} = T(x_1)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x=x_1}.$$

Unfortunately, nothing useful about u can be concluded from this equation. Let's move on to Newton's second law in the u-direction. If the height of the chain is $u(x,t)$, the rate of change of the height with respect to time, the velocity, is given by $\partial u/\partial t = u_t$. Consequently, the rate of change of velocity with respect to time, the acceleration, is $\partial^2 u/\partial t^2 = u_{tt}$. Understand that u_{tt} is the acceleration at a specific point on the chain, x, at time t. To get the force we therefore have to multiply u_{tt} by a tiny bit of mass dm. Mass, of course, is density times length, so this can be written in terms of arc length, s, as $dm = \rho\, ds$. The total force is obtained by integrating $u_{tt}\, dm$ over the mass of the chain. Newton's second law in the u-direction is thus

$$\sum F_u = -T(x_0)\sin\theta_0 + T(x_1)\sin\theta_1 = \int_{\text{mass of chain}} u_{tt}\, dm.$$

Write θ in terms of u using the right triangle in Figure 1 and substitute $dm = \rho\, ds$.

$$-T(x_0)\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=x_0} + T(x_1)\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x=x_1} = \int_{\text{length of chain}} \rho u_{tt}\, ds = \int_{x_0}^{x_1} \rho u_{tt}\sqrt{1+u_x^2}\, dx$$

This equation can be simplified if we make the assumption that u, and therefore u_x, remains small for all x and t. The binomial theorem tells us that

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u_x^4 + \cdots .$$

Compared to 1, u_x^2 and all higher powers of u_x can be considered negligible. Approximating the square root terms as 1, the equation of motion simplifies to

$$T(x_1)u_x(x_1,t) - T(x_0)u_x(x_0,t) \approx \int_{x_0}^{x_1} \rho u_{tt}\,dx.$$

According to the fundamental theorem of calculus,

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

so the left side can be written as

$$T(x_1)u_x(x_1,t) - T(x_0)u_x(x_0,t) = \int_{x_0}^{x_1} \frac{\partial}{\partial x}[T(x)u_x]\,dx.$$

Hence,

$$\int_{x_0}^{x_1} \frac{\partial}{\partial x}[T(x)u_x]\,dx = \int_{x_0}^{x_1} \rho u_{tt}\,dx.$$

Thus, the integrands must be equal to each other.

$$\frac{\partial}{\partial x}[T(x)u_x] = \rho u_{tt}$$

Because the tension is equal to the weight of the chain below it, we can say that

$$T(x) = \rho s(x) g,$$

where $s(x)$ is the arc length of the chain from x to the end of the chain l.

$$T(x) = \int_x^l \rho g \sqrt{1+u_x^2}\,dx,$$

but since the square root term can be approximated as 1, $T(x)$ simplifies to

$$T(x) = \int_x^l \rho g\,dx$$
$$T(x) = \rho g(l-x).$$

Substituting $T(x)$ into the equation of motion gives

$$\frac{\partial}{\partial x}[\rho g(l-x)u_x] = \rho u_{tt}.$$

We assume that the mass density is constant, so it can be pulled out of the derivative and be cancelled from both sides. The equation of motion for a hanging chain is therefore

$$\frac{\partial}{\partial x}[g(l-x)u_x] = u_{tt}.$$

The Differential Formulation

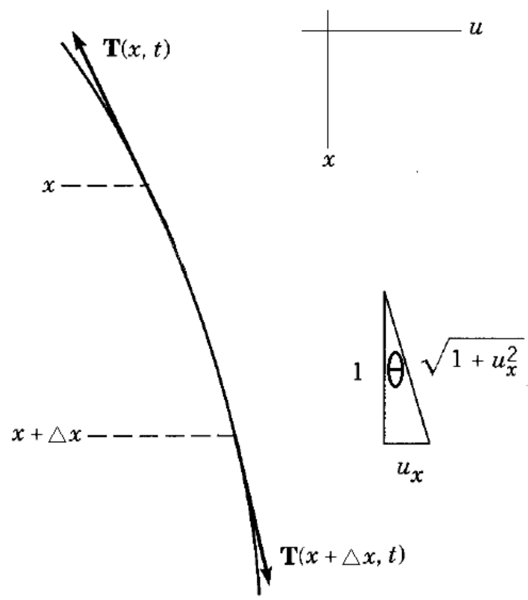

Figure 2: Schematic of the chain (integral formulation).

In order to derive the equation of motion, we will invoke Newton's second law, which states that the sum of the forces acting on a body is equal to its mass times its acceleration. Mathematically this is written as

$$\sum \mathbf{F} = m\mathbf{a}.$$

Note that this is a vector equation; in other words, there is a separate equation for each component of force and corresponding component of acceleration. For this problem we will choose the coordinate system as shown in the figure, so there are two equations of significance.

$$\sum F_x = ma_x$$
$$\sum F_u = ma_u$$

There are two forces acting on this string, $\mathbf{T}$ at x and $\mathbf{T}$ at $x + \Delta x$. Because the tension only depends on the weight of the string below it, it is a function of x only. The motion of the chain is entirely in the u-direction, which means there is no vertical component of acceleration ($a_x = 0$). The tensions at x and $x + \Delta x$ have components in both the x-direction and u-direction and have to be resolved using cosine and sine, respectively.

Component of $\mathbf{T}$ in x-direction at x:	$-T(x)\cos\theta_x$
Component of $\mathbf{T}$ in x-direction at $x + \Delta x$:	$+T(x + \Delta x)\cos\theta_{x+\Delta x}$
Component of $\mathbf{T}$ in u-direction at x:	$-T(x)\sin\theta_x$
Component of $\mathbf{T}$ in u-direction at $x + \Delta x$:	$+T(x + \Delta x)\sin\theta_{x+\Delta x}$,

where θ_x and $\theta_{x+\Delta x}$ are the angles between the vectors and the x-axis at x and $x + \Delta x$, respectively. To determine θ it is necessary to note that $\tan\theta$ is equal to rise over run, the slope.

If the height of the string is $u(x,t)$, the slope is given by $\partial u/\partial x = u_x$. As shown in Figure 2, the hypotenuse can be determined using Pythagorean's theorem. And now $\cos\theta$ can be written in terms of u. Newton's second law in the x-direction is thus

$$\sum F_x = -T(x)\cos\theta_x + T(x+\Delta x)\cos\theta_{x+\Delta x} = ma_x = 0$$

$$-T(x)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_x + T(x+\Delta x)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x+\Delta x} = 0.$$

Hence,

$$T(x)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_x = T(x+\Delta x)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x+\Delta x}.$$

Unfortunately, nothing useful about u can be concluded from this equation. Let's move on to Newton's second law in the u-direction. If the height of the chain is $u(x,t)$, the rate of change of the height with respect to time, the velocity, is given by $\partial u/\partial t = u_t$. Consequently, the rate of change of velocity with respect to time, the acceleration, is $\partial^2 u/\partial t^2 = u_{tt}$. Understand that u_{tt} is the acceleration at a specific point on the chain, x, at time t. To get the force we therefore have to multiply u_{tt} by a tiny bit of mass Δm. Mass, of course, is density times length, so this can be written in terms of arc length, s, as $dm = \rho\,\Delta s$. Newton's second law in the u-direction is thus

$$\sum F_u = -T(x)\sin\theta_x + T(x+\Delta x)\sin\theta_{x+\Delta x} = u_{tt}\,\Delta m.$$

Write θ in terms of u using the right triangle in Figure 2 and substitute $\Delta m = \rho\,\Delta s$.

$$-T(x)\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_x + T(x+\Delta x)\left.\frac{u_x}{\sqrt{1+u_x^2}}\right|_{x+\Delta x} = \rho u_{tt}\,\Delta s = \rho u_{tt}\sqrt{1+u_x^2}\,\Delta x$$

This equation can be simplified if we make the assumption that u, and therefore u_x, remains small for all x and t. The binomial theorem tells us that

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u_x^4 + \cdots.$$

Compared to 1, u_x^2 and all higher powers of u_x can be considered negligible. Approximating the square root terms as 1, the equation of motion becomes

$$T(x+\Delta x)u_x(x+\Delta x,t) - T(x)u_x(x,t) \approx \rho u_{tt}\,\Delta x.$$

Divide both sides by Δx.

$$\frac{T(x+\Delta x)u_x(x+\Delta x,t) - T(x)u_x(x,t)}{\Delta x} = \rho u_{tt}$$

Now take the limit of both sides as $\Delta x \to 0$.

$$\lim_{\Delta x\to 0}\frac{T(x+\Delta x)u_x(x+\Delta x,t) - T(x)u_x(x,t)}{\Delta x} = \lim_{\Delta x\to 0}\rho u_{tt}$$

According to the definition of the derivative,

$$\frac{df}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Hence,

$$\frac{\partial}{\partial x}[T(x)u_x] = \rho u_{tt}$$

Because the tension is equal to the weight of the chain below it, we can say that

$$T(x) = \rho s(x) g,$$

where $s(x)$ is the arc length of the chain from x to the end of the chain l.

$$T(x) = \int_x^l \rho g \sqrt{1 + u_x^2}\, dx,$$

but since the square root term can be approximated as 1, $T(x)$ simplifies to

$$\begin{aligned} T(x) &= \int_x^l \rho g\, dx \\ T(x) &= \rho g(l - x). \end{aligned}$$

Substituting $T(x)$ into the equation of motion gives

$$\frac{\partial}{\partial x}[\rho g(l - x)u_x] = \rho u_{tt}.$$

We assume that the mass density is constant, so it can be pulled out of the derivative and be cancelled from both sides. The equation of motion for a hanging chain is therefore

$$\frac{\partial}{\partial x}[g(l - x)u_x] = u_{tt}.$$