Exercise 3

On the sides of a thin rod, heat exchange takes place (obeying Newton's law of cooling—flux proportional to temperature difference) with a medium of constant temperature T_0. What is the equation satisfied by the temperature $u(x,t)$, neglecting its variation across the rod?

Solution

There are two ways (among others) to go about this problem. One is the integral formulation, where we consider the heat flowing into and out of a finite portion of the rod. The other is the differential formulation, where we consider the heat flowing into and out of an infinitesimal element of the rod. In both cases we come to the same governing PDE, so use whichever you prefer.

Two modes of heat transfer are at work in this problem: conduction (axially along the rod's interior) and convection (on the sides of the rod). To find the governing PDE for the temperature we have to consider the law of conservation of energy, which says that energy can neither be created nor destroyed. If there's a certain amount of heat entering a portion of the rod and there's less heat coming out of it, then there must be heat accumulating within it. Mathematically this is written as

$$\frac{dq}{dt} = \text{rate of heat in} - \text{rate of heat out}, \tag{1}$$

where q denotes the amount of heat and dq/dt represents how fast heat is accumulating. The heat flux due to conduction ϕ is defined as the amount of heat flowing axially per unit time per unit area. Therefore, if we multiply the heat flux by the area of the rod's cross section, we get the rate of heat that is flowing through it.

The Integral Formulation

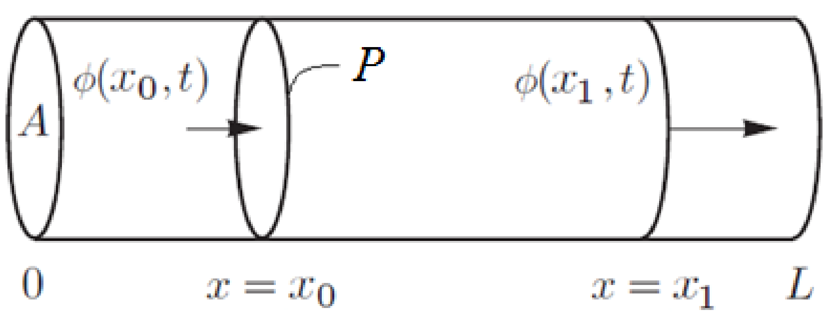

Figure 1: Schematic of the thin rod (integral formulation). Do note that although the rod here has a circular cross section, the following analysis is for a general cross section with area A and perimeter P.

If we consider the portion of the rod between $x = x_0$ and $x = x_1$ as shown in Figure 1, heat is flowing into the portion at $x = x_0$ and flowing out of it at $x = x_1$, as indicated by the arrows.

$$\begin{aligned} \text{Rate of heat flow in:} \quad & \phi(x_0, t)A \\ \text{Rate of heat flow out:} \quad & \phi(x_1, t)A \end{aligned}$$

These are the rates for conduction. Now we will find the rate of heat flow due to convection by using Newton's law of cooling, which says that heat flux due to convection ψ is proportional to the temperature difference between the rod and the environment it's in.

$$\psi \propto T - T_0,$$

where T is the temperature of the rod (a function of x and t) and T_0 is the ambient temperature (a constant). To change this to an equation we can use, we introduce a constant of proportionality h, which is known as the convection heat transfer coefficient.

$$\psi = h(T - T_0)$$

If the rod is hot compared to the environment (i.e. $T > T_0$), heat will flow out of the rod. Conversely, if the rod is cold (i.e. $T < T_0$), heat will flow into it. As mentioned before, to get the rate of heat flow we multiply the heat flux by the area. Note, however, that this area is not the cross-sectional area A we used before for conduction, but rather it is the surface area that is exposed to the environment. Because different parts of the bar are at different temperatures, we have to multiply the heat flux by the perimeter of the cross section and a little bit of distance dx to get a little bit of surface area. To get the total rate of heat flow we integrate over the portion of the rod from $x = x_0$ to $x = x_1$.

$$\text{Rate of heat flow out:} \quad \int_{x_0}^{x_1} \psi(x, t)P\, dx = \int_{x_0}^{x_1} hP(T - T_0)\, dx$$

This is the rate for convection. Now we can substitute these results into the law of conservation of energy (1).

$$\frac{dq}{dt} = \phi(x_0, t)A - \phi(x_1, t)A - \int_{x_0}^{x_1} hP(T - T_0)\, dx$$

The relationship between the heat q and temperature T is $dq = mc\, dT$, where m is the mass of the portion of the rod and c is the specific heat (a measure of how hard it is to change the temperature) of the rod. Thus,

$$\frac{dq}{dt} = mc\frac{\partial T}{\partial t}.$$

m is the product of mass density ρ (mass per unit volume) and $A\, dx$. Since $\partial T/\partial t$ is different at different parts of the rod, we have to integrate from $x = x_0$ to $x = x_1$ to get the total dq/dt. Plugging this in to the left side gives us

$$\int_{x_0}^{x_1} \rho Ac\frac{\partial T}{\partial t}\, dx = -[\phi(x_1, t)A - \phi(x_0, t)A] - \int_{x_0}^{x_1} hP(T - T_0)\, dx.$$

According to the fundamental theorem of calculus,

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

so we can write the part in square brackets as

$$\phi(x_1,t)A - \phi(x_0,t)A = \int_{x_0}^{x_1} \frac{\partial}{\partial x}[\phi(x,t)A]\,dx.$$

The equation for the temperature becomes

$$\int_{x_0}^{x_1} \rho Ac\frac{\partial T}{\partial t}\,dx = -\int_{x_0}^{x_1} \frac{\partial}{\partial x}[\phi(x,t)A]\,dx - \int_{x_0}^{x_1} hP(T-T_0)\,dx.$$

Combine the two integrals on the right into one.

$$\int_{x_0}^{x_1} \rho Ac\frac{\partial T}{\partial t}\,dx = \int_{x_0}^{x_1} \left\{-\frac{\partial}{\partial x}[\phi(x,t)A] - hP(T-T_0)\right\} dx$$

Hence, the integrands must be equal to each other.

$$\rho Ac\frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}[\phi(x,t)A] - hP(T-T_0)$$

The final step is to express the heat flux due to conduction ϕ in terms of the temperature, and this is done using Fourier's law of conduction, which states that heat flux is proportional to the temperature gradient.

$$\phi \propto \frac{\partial T}{\partial x}$$

To make this into an equation, we introduce a constant of proportionality κ, which is known as thermal conductivity.

$$\phi = -\kappa\frac{\partial T}{\partial x}$$

We include the minus sign to indicate that heat travels down a temperature gradient, that is, from hot to cold regions. Plugging this expression in for ϕ into the equation for temperature, it becomes

$$\rho Ac\frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}\left(-\kappa\frac{\partial T}{\partial x}A\right) - hP(T-T_0)$$

Since κ and A are constant, we can pull them out of the derivative.

$$\rho Ac\frac{\partial T}{\partial t} = \kappa A\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) - hP(T-T_0)$$

Divide both sides by ρAc.

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c}\frac{\partial^2 T}{\partial x^2} - \frac{hP}{\rho Ac}(T-T_0)$$

Therefore,

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} - \frac{hP}{\rho Ac}(T-T_0),$$

where $k = \kappa/\rho c$ is another constant known as the thermal diffusivity. Since the problem statement wants $u(x,t)$ to represent the temperature, let $u(x,t) = T(x,t)$.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \frac{hP}{\rho Ac}(u-T_0)$$

NOTE: In the answer at the back of the book, Mr. Strauss uses μ for h and calls it "conductance," but know that heat transfer texts generally use h and call it the *convection* heat transfer coefficient.

The Differential Formulation

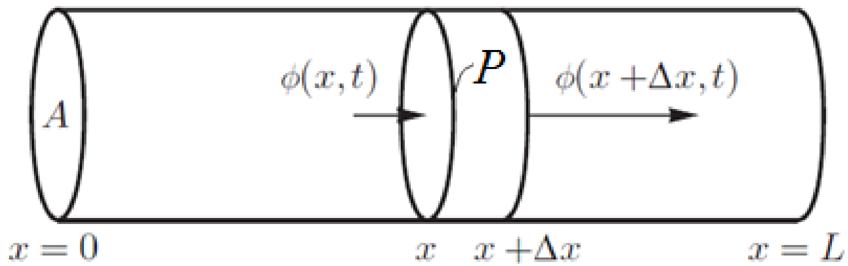

Figure 2: Schematic of the thin rod (differential formulation). Do note that although the rod here has a circular cross section, the following analysis is for a general cross section with area A and perimeter P.

If we consider the portion of the rod between x and $x+\Delta x$ as shown in Figure 1, heat is flowing into the portion at x and flowing out of it at $x+\Delta x$, as indicated by the arrows.

$$\begin{aligned} \text{Rate of heat flow in:} &\quad \phi(x,t)A \\ \text{Rate of heat flow out:} &\quad \phi(x+\Delta x,t)A \end{aligned}$$

These are the rates for conduction. Now we will find the rate of heat flow due to convection by using Newton's law of cooling, which says that heat flux due to convection ψ is proportional to the temperature difference between the rod and the environment it's in.

$$\psi \propto T - T_0,$$

where T is the temperature of the rod (a function of x and t) and T_0 is the ambient temperature (a constant). To change this to an equation we can use, we introduce a constant of proportionality h, which is known as the convection heat transfer coefficient.

$$\psi = h(T - T_0)$$

If the rod is hot compared to the environment (i.e. $T > T_0$), heat will flow out of the rod. Conversely, if the rod is cold (i.e. $T < T_0$), heat will flow into it. As mentioned before, to get the rate of heat flow we multiply the heat flux by the area. Note, however, that this area is not the cross-sectional area A we used before for conduction, but rather it is the surface area that is exposed to the environment, which is the perimeter P times distance Δx.

$$\text{Rate of heat flow out:} \quad \psi(x,t)P\,\Delta x = hP(T - T_0)\,\Delta x$$

This is the rate for convection. Now we can substitute these results into the law of conservation of energy (1).

$$\frac{dq}{dt} = \phi(x,t)A - \phi(x+\Delta x,t)A - hP(T-T_0)\,\Delta x$$

The relationship between the heat q and temperature T is $dq = mc\,dT$, where m is the mass of the portion of the rod and c is the specific heat (a measure of how hard it is to change the temperature) of the rod. Thus,

$$\frac{dq}{dt} = mc\frac{\partial T}{\partial t}.$$

m is the product of mass density ρ (mass per unit volume) and $A\,\Delta x$. Plugging this in to the left side gives us

$$\rho Ac\frac{\partial T}{\partial t}\,\Delta x = -[\phi(x+\Delta x,t)A - \phi(x,t)A] - hP(T-T_0)\,\Delta x.$$

Divide both sides by Δx.

$$\rho Ac\frac{\partial T}{\partial t} = -\frac{\phi(x+\Delta x,t)A - \phi(x,t)A}{\Delta x} - hP(T-T_0)$$

Now let $\Delta x \to 0$.

$$\rho Ac\frac{\partial T}{\partial t} = -\lim_{\Delta x\to 0}\frac{\phi(x+\Delta x,t)A - \phi(x,t)A}{\Delta x} - hP(T-T_0)$$

According to the definition of the derivative,

$$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

Hence, the equation for the temperature becomes

$$\rho Ac\frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}[\phi(x,t)A] - hP(T-T_0).$$

The final step is to express the heat flux due to conduction ϕ in terms of the temperature, and this is done using Fourier's law of conduction, which states that heat flux is proportional to the temperature gradient.

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Since κ and A are constant, we can pull them out of the derivative.

$$\rho Ac\frac{\partial T}{\partial t} = \kappa A\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) - hP(T-T_0)$$

Divide both sides by ρAc.

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho c}\frac{\partial^2 T}{\partial x^2} - \frac{hP}{\rho Ac}(T-T_0)$$

Therefore,

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} - \frac{hP}{\rho Ac}(T-T_0),$$

where $k = \kappa/\rho c$ is another constant known as the thermal diffusivity. Since the problem statement wants $u(x,t)$ to represent the temperature, let $u(x,t) = T(x,t)$.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \frac{hP}{\rho Ac}(u-T_0)$$

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