Exercise 6

Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.

Solution

The three-dimensional heat equation is given as

$$\rho c u_t = \nabla \cdot (\kappa \nabla u).$$

If we assume the circular cylinder is homogeneous, then κ does not depend on the spatial coordinates and can be pulled out in front.

$$c\rho u_t = \kappa \nabla \cdot \nabla u = \kappa \nabla^2 u$$

Thus,

$$u_t = k\nabla^2 u, \tag{1}$$

where $k = \kappa / c\rho$. In cylindrical coordinates, the laplacian operator is defined as follows.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

But since the only spatial coordinate u depends on is r, $\partial^2 u/\partial \theta^2 = 0$ and $\partial^2 u/\partial z^2 = 0$.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{=0}$$

The laplacian simplifies to

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Substituting this expression for $\nabla^2 u$ into (1) gives us

$$u_t = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]$$

$$u_t = \frac{k}{r}\left(\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right)$$

$$u_t = k\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right).$$

Therefore,

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r\right).$$