Exercise 7

Solve Exercise 6 in a ball except that the temperature depends only on the spherical coordinate $\sqrt{x^2+y^2+z^2}$. Derive the equation $u_t = k(u_{rr} + 2u_r/r)$.

Solution

The three-dimensional heat equation is given as

$$\rho c u_t = \nabla \cdot (\kappa \nabla u).$$

If we assume the ball is homogeneous, then κ does not depend on the spatial coordinates and can be pulled out in front.

$$c\rho u_t = \kappa \nabla \cdot \nabla u = \kappa \nabla^2 u$$

Thus,

$$u_t = k\nabla^2 u, \tag{1}$$

where $k = \kappa/c\rho$. In spherical coordinates, the laplacian operator is defined as follows.

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

But since the only spatial coordinate u depends on is r, $\partial u/\partial\theta = 0$ and $\partial^2 u/\partial\phi^2 = 0$.

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right)}_{=0} + \underbrace{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}}_{=0}$$

The laplacian simplifies to

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

Substituting this expression for $\nabla^2 u$ into (1) gives us

$$\begin{aligned} u_t &= k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)\right] \\ u_t &= \frac{k}{r^2}\left(2r\frac{\partial u}{\partial r} + r^2 \frac{\partial^2 u}{\partial r^2}\right) \\ u_t &= k\left(\frac{2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right). \end{aligned}$$

Therefore,

$$u_t = k\left(u_{rr} + \frac{2}{r}u_r\right).$$