Exercise 2

(a) Show that the temperature of a metal rod, insulated at the end $x = 0$, satisfies the boundary condition $\partial u/\partial x = 0$. (Use Fourier's law.)

(b) Do the same for the diffusion of gas along a tube that is closed off at the end $x = 0$. (Use Fick's law.)

(c) Show that the three-dimensional version of (a) (insulated solid) or (b) (impermeable container) leads to the boundary condition $\partial u/\partial n = 0$.

Solution

Part (a)

Fourier's law says that the heat flux by conduction is proportional to the temperature gradient. In one dimension this can be written mathematically as

$$\underbrace{q}_{\text{heat flux}} \quad \underbrace{\propto}_{\text{proportional to}} \quad - \underbrace{\frac{\partial T}{\partial x}}_{\text{temp. gradient}}.$$

To change this to an equation we can use, we have to introduce a constant of proportionality κ. For Fourier's law in particular, it is called the thermal conductivity. The minus sign indicates that heat always moves in the direction of decreasing temperature.

$$q = -\kappa \frac{\partial T}{\partial x}$$

Because the metal rod is insulated at $x = 0$, this means that no heat can enter or leave at $x = 0$, that is, $q = 0$.

$$0 = -\kappa \frac{\partial T}{\partial x}(x = 0, t)$$

Therefore, the boundary condition is

$$\frac{\partial T}{\partial x}(0, t) = 0.$$

Part (b)

Fick's first law says that the molecular mass flux is proportional to the concentration gradient. In one dimension this can be written mathematically as

$$\underbrace{j}_{\text{mass flux}} \quad \underbrace{\propto}_{\text{proportional to}} \quad - \underbrace{\frac{\partial c}{\partial x}}_{\text{conc. gradient}}.$$

To change this to an equation we can use, we have to introduce a constant of proportionality D. For Fick's first law in particular, it is called the diffusion constant. The minus sign indicates that mass always moves down a concentration gradient.

$$j = -D \frac{\partial c}{\partial x}$$

Because the tube is closed off at $x = 0$, this means that no mass can enter or leave at $x = 0$, that is, $j = 0$.

$$0 = -D\frac{\partial c}{\partial x}(x = 0, t)$$

Therefore, the boundary condition is

$$\frac{\partial c}{\partial x}(0, t) = 0.$$

Part (c)

Fourier's law in three dimensions is

$$\mathbf{q} = -\kappa \nabla T.$$

Because the boundary is insulated, that means no heat can enter or leave, which means $\mathbf{q} = \mathbf{0}$.

$$\mathbf{0} = -\kappa \nabla T$$

$$\nabla T = \mathbf{0}$$

If we take an outward unit vector that is perpendicular to the solid's surface, $\hat{\mathbf{n}}$, and dot it with both sides, we obtain the boundary condition.

$$\hat{\mathbf{n}} \cdot \nabla T = \hat{\mathbf{n}} \cdot \mathbf{0}$$

Since $\hat{\mathbf{n}} \cdot \nabla T = \partial T/\partial n$, the boundary condition is

$$\frac{\partial T}{\partial n} = 0.$$

The same is true regarding mass transport. Fick's first law in three dimensions is

$$\mathbf{j} = -D\nabla c$$

Because the boundary is impermeable, no mass can enter or leave it. Hence, $\mathbf{j} = \mathbf{0}$.

$$\mathbf{0} = -D\nabla c$$

$$\nabla c = \mathbf{0}$$

If we take an outward unit vector that is perpendicular to the solid's surface, $\hat{\mathbf{n}}$, and dot it with both sides, we obtain the boundary condition.

$$\hat{\mathbf{n}} \cdot \nabla c = \hat{\mathbf{n}} \cdot \mathbf{0}$$

Since $\hat{\mathbf{n}} \cdot \nabla c = \partial c/\partial n$, the boundary condition is

$$\frac{\partial c}{\partial n} = 0.$$