Exercise 3

A homogeneous body occupying the solid region D is completely insulated. Its initial temperature is $f(\mathbf{x})$. Find the steady-state temperature that it reaches after a long time. (*Hint:* No heat is gained or lost.)

Solution

The three-dimensional heat equation is

$$\rho c u_t = \nabla \cdot (\kappa \nabla u).$$

If we integrate both sides over the volume of the solid, we get

$$\iiint_D \rho c u_t \, dV = \iiint_D \nabla \cdot (\kappa \nabla u) \, dV. \tag{1}$$

Recall from calculus that the divergence theorem says if we have a vector field $\mathbf{F}$, then

$$\iiint_D \nabla \cdot \mathbf{F} \, d\mathbf{x} = \iint_{\text{bdy } D} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS,$$

where $\hat{\mathbf{n}}$ is an outward unit vector perpendicular to the surface. Applying this to the right side of (1), we get

$$\iiint_D \rho c u_t \, dV = \iint_{\text{bdy } D} \kappa \nabla u \cdot \hat{\mathbf{n}} \, dS.$$

Since $\nabla u \cdot \hat{\mathbf{n}} = \partial u / \partial n$, we can write this surface integral as

$$\iiint_D \rho c u_t \, dV = \iint_{\text{bdy } D} \kappa \frac{\partial u}{\partial n} \, dS.$$

Because the solid is completely insulated, no heat can enter or leave the surface, which means $\partial u / \partial n = 0$. Hence,

$$\iiint_D \rho c u_t \, dV = \iint_{\text{bdy } D} \kappa \underbrace{\frac{\partial u}{\partial n}}_{=0} \, dS = 0.$$

Now we work with the left side of the equation.

$$\iiint_D \rho c \frac{\partial u}{\partial t} \, dV = 0$$

$$\frac{d}{dt} \iiint_D \rho c u \, dV = 0$$

This equation indicates that the total thermal energy in the solid,

$$\iiint_D \rho c u \, dV,$$

is constant in time. This makes sense because the boundary is insulated—no heat enters or leaves it. Whatever the initial temperature distribution of the solid is, we expect that after a long time the temperature will be the same throughout. If $u(\mathbf{x},0) = f(\mathbf{x})$ is the initial temperature distribution and $u(\mathbf{x},t) = \bar{T}$ is the constant temperature the ball tends to as $t \to \infty$, then

$$\iiint_D \rho c f(\mathbf{x})\,dV = \iiint_D \rho c \bar{T}\,dV.$$

Since the heat capacity and the mass density are constant, c and ρ come out in front of the integral and cancel from both sides. Also, because $\bar{T}$ is constant, it comes out of the integral too.

$$\iiint_D f(\mathbf{x})\,dV = \bar{T} \iiint_D dV$$

Therefore,

$$\bar{T} = \frac{\iiint_D f(\mathbf{x})\,dV}{\iiint_D dV}.$$

The eventual constant (steady-state) temperature is the average of the initial temperature distribution over the volume of the solid.