Exercise 4

A rod occupying the interval $0 \leq x \leq l$ is subject to the heat source $f(x) = 0$ for $0 < x < \frac{l}{2}$, and $f(x) = H$ for $\frac{l}{2} < x < l$ where $H > 0$. The rod has physical constants $c = \rho = \kappa = 1$, and its ends are kept at zero temperature.

(a) Find the steady-state temperature of the rod.

(b) Which point is the hottest, and what is the temperature there?

Solution

Part (a)

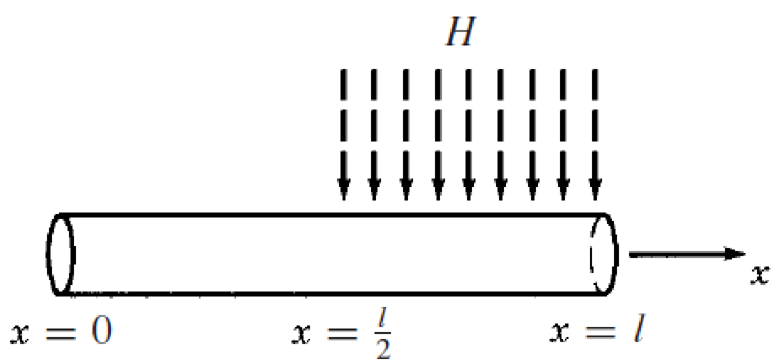

Figure 1: Schematic of the rod with the given heat source.

The governing PDE for the temperature in a bar with a heat source is given by

$$T_t = \nabla \cdot (k\nabla T) + f(x, t).$$

The steady state of the temperature is reached as $t \to \infty$. In other words, after a long time has passed, the temperature will no longer vary as a function of time. Hence, $T_t = 0$. Because all constants are unity, the heat source varies only with position, and the rod is one-dimensional, the heat equation reduces to

$$0 = \frac{d^2T}{dx^2} + f(x)$$

in the steady state. That is,

$$\begin{cases} 0 = \dfrac{d^2T}{dx^2} & 0 < x < \dfrac{l}{2} \\ 0 = \dfrac{d^2T}{dx^2} + H & \dfrac{l}{2} < x < l \end{cases}.$$

These two ODEs can be solved by straightforward integration.

$$T(x) = \begin{cases} Ax + B & 0 < x < \dfrac{l}{2} \\ -\dfrac{1}{2}Hx^2 + Cx + D & \dfrac{l}{2} < x < l \end{cases}$$

We're told that the ends of the rod are kept at zero temperature, so the boundary conditions are $T(0, t) = 0$ and $T(l, t) = 0$ for all time. In the steady state, though, we use $T(0) = 0$ and $T(l) = 0$.

We have four constants of integration; therefore, we need two more conditions to determine the solution. These two additional conditions come from physical grounds: The temperature and the heat flux must be continuous at the point, $x = l/2$, where the heat source changes. Thus,

$$\lim_{x\to\frac{l}{2}-} T(x) = \lim_{x\to\frac{l}{2}+} T(x)$$
$$\lim_{x\to\frac{l}{2}-} -\kappa\frac{dT}{dx} = \lim_{x\to\frac{l}{2}+} -\kappa\frac{dT}{dx}.$$

These four conditions mean that

$$\begin{aligned}
T(0) = 0 \quad &\rightarrow \quad B = 0 \\
T(l) = 0 \quad &\rightarrow \quad D = \frac{1}{2}Hl^2 - Cl \\
T\left(\frac{l}{2}-\right) = T\left(\frac{l}{2}+\right) \quad &\rightarrow \quad \frac{Al}{2} = \frac{l}{8}(3Hl - 4C) \\
\frac{dT}{dx}\left(\frac{l}{2}-\right) = \frac{dT}{dx}\left(\frac{l}{2}+\right) \quad &\rightarrow \quad A = C - \frac{Hl}{2}.
\end{aligned}$$

Solving this system of equations for the constants, plugging in the values, and simplifying, we arrive at the steady state temperature.

$$T(x) = \begin{cases} \frac{Hl}{8}x & 0 < x < \frac{l}{2} \\ -\frac{H}{8}(l-4x)(l-x) & \frac{l}{2} < x < l \end{cases}$$

Setting $l = H = 1$, we can graph this to get an idea of what the temperature looks like in the steady state.

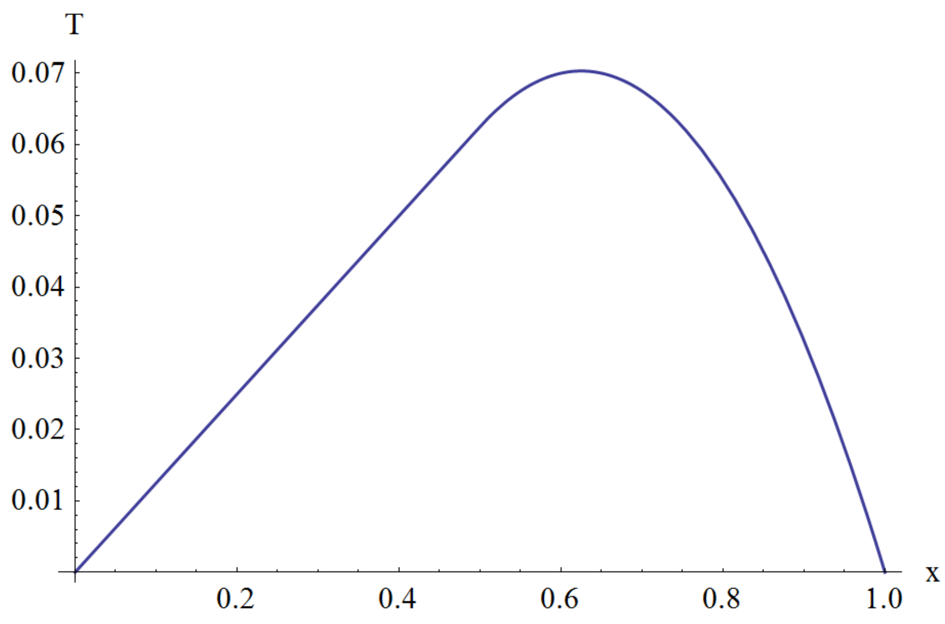

Figure 2: Plot of $T(x)$ vs. x.

Part (b)

The maximum temperature occurs where the first derivative of the steady-state temperature equals zero. Taking the derivative with respect to x of the function defined for $\frac{l}{2} < x < l$ gives

$$H\left(\frac{5l}{8} - x\right).$$

By inspection we can see that if

$$x = \frac{5l}{8},$$

the first derivative vanishes, and that means this is where the hottest temperature occurs in the bar. Evaluating $T(x)$ at this point tells us what the hottest temperature is.

$$T\left(x = \frac{5l}{8}\right) = \frac{9Hl^2}{128}$$

$9/128 \approx 0.0703$ and $5/8 = 0.625$, so the graph is consistent with our conclusions.