Exercise 7

In linearized gas dynamics (sound), verify the following.

(a) If $\operatorname{curl}\mathbf{v} = \mathbf{0}$ at $t = 0$, then $\operatorname{curl}\mathbf{v} = \mathbf{0}$ at all later times.

(b) Each component of $\mathbf{v}$ and ρ satisfies the wave equation.

Solution

The linearized equations for the velocity $\mathbf{v}$ and density ρ are given as

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{c_0^2}{\rho_0}\operatorname{grad}\rho = 0 \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div}\mathbf{v} = 0. \tag{2}$$

Part (a)

We want to show that $\operatorname{curl}\mathbf{v}$ is constant for all time, that is,

$$\frac{d}{dt}\operatorname{curl}\mathbf{v} = 0.$$

$$\frac{d}{dt}\operatorname{curl}\mathbf{v} = \operatorname{curl}\frac{\partial \mathbf{v}}{\partial t}$$

Solve (1) for $\partial\mathbf{v}/\partial t$ and substitute it.

$$\frac{d}{dt}\operatorname{curl}\mathbf{v} = \operatorname{curl}\left(-\frac{c_0^2}{\rho_0}\operatorname{grad}\rho\right)$$

$$\frac{d}{dt}\operatorname{curl}\mathbf{v} = -\frac{c_0^2}{\rho_0}\operatorname{curl}(\operatorname{grad}\rho)$$

But since the curl of the gradient of any vector function is $\mathbf{0}$, the right side vanishes.

$$\frac{d}{dt}\operatorname{curl}\mathbf{v} = \mathbf{0}$$

Therefore, $\operatorname{curl}\mathbf{v}$ remains constant in time. If it is equal to $\mathbf{0}$ at $t = 0$, then it is equal to $\mathbf{0}$ at all later times.

Part (b)

We start by differentiating both sides of (1) and (2) with respect to time.

$$\begin{cases} \dfrac{\partial^2 \mathbf{v}}{\partial t^2} + \dfrac{c_0^2}{\rho_0} \dfrac{d}{dt} \operatorname{grad} \rho = 0 \\ \dfrac{\partial^2 \rho}{\partial t^2} + \rho_0 \dfrac{d}{dt} \operatorname{div} \mathbf{v} = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \mathbf{v}}{\partial t^2} + \dfrac{c_0^2}{\rho_0} \operatorname{grad} \dfrac{\partial \rho}{\partial t} = 0 \\ \dfrac{\partial^2 \rho}{\partial t^2} + \rho_0 \operatorname{div} \dfrac{\partial \mathbf{v}}{\partial t} = 0 \end{cases}$$

Now substitute (2) into the first equation and (1) into the second equation.

$$\begin{cases} \dfrac{\partial^2 \mathbf{v}}{\partial t^2} + \dfrac{c_0^2}{\rho_0} \operatorname{grad} (-\rho_0 \operatorname{div} \mathbf{v}) = 0 \\ \dfrac{\partial^2 \rho}{\partial t^2} + \rho_0 \operatorname{div} \left(-\dfrac{c_0^2}{\rho_0} \operatorname{grad} \rho \right) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \mathbf{v}}{\partial t^2} - c_0^2 \operatorname{grad} (\operatorname{div} \mathbf{v}) = 0 \\ \dfrac{\partial^2 \rho}{\partial t^2} - c_0^2 \operatorname{div} (\operatorname{grad} \rho) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \mathbf{v}}{\partial t^2} - c_0^2 \Delta \mathbf{v} = 0 \\ \dfrac{\partial^2 \rho}{\partial t^2} - c_0^2 \Delta \rho = 0 \end{cases}$$

The first equation implies that

$$\begin{cases} \dfrac{\partial^2 v_x}{\partial t^2} - c_0^2 \Delta v_x = 0 \\ \dfrac{\partial^2 v_y}{\partial t^2} - c_0^2 \Delta v_y = 0 \\ \dfrac{\partial^2 v_z}{\partial t^2} - c_0^2 \Delta v_z = 0 \end{cases} .$$

Therefore, each component of $\mathbf{v}$ and ρ satisfy the wave equation.