Exercise 2

Consider the problem

$$u''(x) + u'(x) = f(x)$$

$$u'(0) = u(0) = \frac{1}{2}\big[u'(l) + u(l)\big],$$

with $f(x)$ a given function.

(a) Is the solution *unique*? Explain.

(b) Does a solution necessarily *exist*, or is there a condition that $f(x)$ must satisfy for existence? Explain.

Solution

Part (a)

Suppose there are two solutions to this boundary value problem, u_1 and u_2. Then they both satisfy the ODE.

$$u_1''(x) + u_1'(x) = f(x)$$
$$u_2''(x) + u_2'(x) = f(x)$$

If there is a unique solution to the problem, then u_1 and u_2 must be equal. Subtract the second equation from the first.

$$u_1'' - u_2'' + u_1' - u_2' = 0$$

$$(u_1 - u_2)'' + (u_1 - u_2)' = 0$$

Make the substitution, $w = u_1 - u_2$.

$$w'' + w' = 0$$

This is a linear ODE with constant coefficients, so the solution will be of the form, $w = e^{rx}$.

$$w = e^{rx} \quad \rightarrow \quad \frac{dw}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2w}{dx^2} = r^2e^{rx}$$

Substituting these expressions into the equation gives us

$$r^2e^{rx} + re^{rx} = 0.$$

Dividing both sides by e^{rx} yields a polynomial in r that we can solve.

$$r^2 + r = 0$$

$$r(r+1) = 0$$

$$r = \{-1, 0\}$$

Thus,

$$w(x) = Ae^{-x} + B.$$

Because $w = u_1 - u_2 \neq 0$, the solution to the ODE is not unique.

Part (b)

Start off by integrating both sides of the ODE from 0 to l.

$$u'' + u' = f$$

$$\int_0^l (u'' + u')\,dx = \int_0^l f(x)\,dx$$

$$\int_0^l u''\,dx + \int_0^l u'\,dx = \int_0^l f(x)\,dx$$

$$u'\big|_0^l + u\big|_0^l = \int_0^l f(x)\,dx$$

$$u'(l) - u'(0) + u(l) - u(0) = \int_0^l f(x)\,dx$$

The boundary conditions are

$$u(0) = u'(0) = \frac{1}{2}[u'(l) + u(l)],$$

so if we plug these in to the left side of the equation, we get

$$u'(l) - \frac{1}{2}[u'(l) + u(l)] + u(l) - \frac{1}{2}[u'(l) + u(l)] = \int_0^l f(x)\,dx$$

$$\cancel{u'(l)} - \cancel{u'(l)} + \cancel{u(l)} - \cancel{u(l)} = \int_0^l f(x)\,dx.$$

Therefore, in order for a solution to exist, $f(x)$ must satisfy the following condition.

$$\int_0^l f(x)\,dx = 0$$

That is, the average of f over the length must be 0 for the solution to exist.