Exercise 3

Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

Solution[1]

The ODE can be solved simply by integrating.

$$u(x) = Ax + B$$

Apply the first boundary condition.

$$u'(0) + ku(0) = 0 \quad \rightarrow \quad A + kB = 0 \tag{1}$$

Case +

Here we use the second boundary condition with the plus (+) sign.

$$u'(1) + ku(1) = 0 \quad \rightarrow \quad A + k(A + B) = 0 \tag{2}$$

Solving (1) and (2) for A and B yields

$$\begin{aligned} A &= -kB \\ -k^2 B &= 0. \end{aligned}$$

If $k \neq 0$, then $B = 0$ and $A = 0$, which leads to $u(x) = 0$. If $k = 0$, then B is arbitrary and $A = 0$, which leads to $u(x) = (0)x + B = B$. Therefore,

$$u(x) = \begin{cases} 0 & \text{if } k \neq 0 \\ B & \text{if } k = 0 \end{cases}.$$

Case −

Here we use the second boundary condition with the negative (−) sign.

$$u'(1) - ku(1) = 0 \quad \rightarrow \quad A - k(A + B) = 0 \tag{3}$$

Solving (1) and (3) for A and B yields

$$\begin{aligned} A &= -kB \\ B(k^2 - 2k) &= 0. \end{aligned}$$

If $k \neq 0$ and $k \neq 2$, then $B = 0$ and $A = 0$, which leads to $u(x) = 0$. If $k = 0$, then B is arbitrary and $A = 0$, which leads to $u(x) = (0)x + B = B$. If $k = 2$, then B is arbitrary and $A = -2B$, which leads to $u(x) = (-2B)x + B = B(1 - 2x)$. Therefore,

$$u(x) = \begin{cases} 0 & \text{if } k \neq 0 \text{ and } k \neq 2 \\ B & \text{if } k = 0 \\ B(1 - 2x) & \text{if } k = 2 \end{cases}.$$

[1]Special thanks to L. Baker for letting me know I made a mistake here.