Exercise 5

Consider the equation

$$u_x + yu_y = 0$$

with the boundary condition $u(x, 0) = \phi(x)$.

(a) For $\phi(x) \equiv x$, show that no solution exists.

(b) For $\phi(x) \equiv 1$, show that there are many solutions.

Solution

On the paths defined by

$$\frac{dy}{dx} = y, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(x, \xi) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Solving (1) by separation of variables gives

$$\begin{aligned}
\frac{dy}{y} &= dx \\
\ln|y| &= x + C \\
|y| &= e^{x+C} \\
y &= \pm e^C e^x \\
y &= \xi e^x.
\end{aligned}$$

Solving for ξ gives

$$\xi = ye^{-x}.$$

Therefore,

$$u(x, y) = f\left(ye^{-x}\right).$$

We can check that this is the solution of the PDE.

$$\begin{aligned}
u_x &= -ye^{-x} f' \\
u_y &= e^{-x} f'
\end{aligned}$$

$u_x + yu_y = 0$, so this is the correct solution. Shown below in Figure 1 are the characteristic curves in the xy-plane for various values of ξ along with the line $y = 0$ (where the boundary condition is defined). Note that because the data curve, $y = 0$, only intersects the $\xi = 0$ characteristic, the solution is only defined for $y = 0$ and all x.

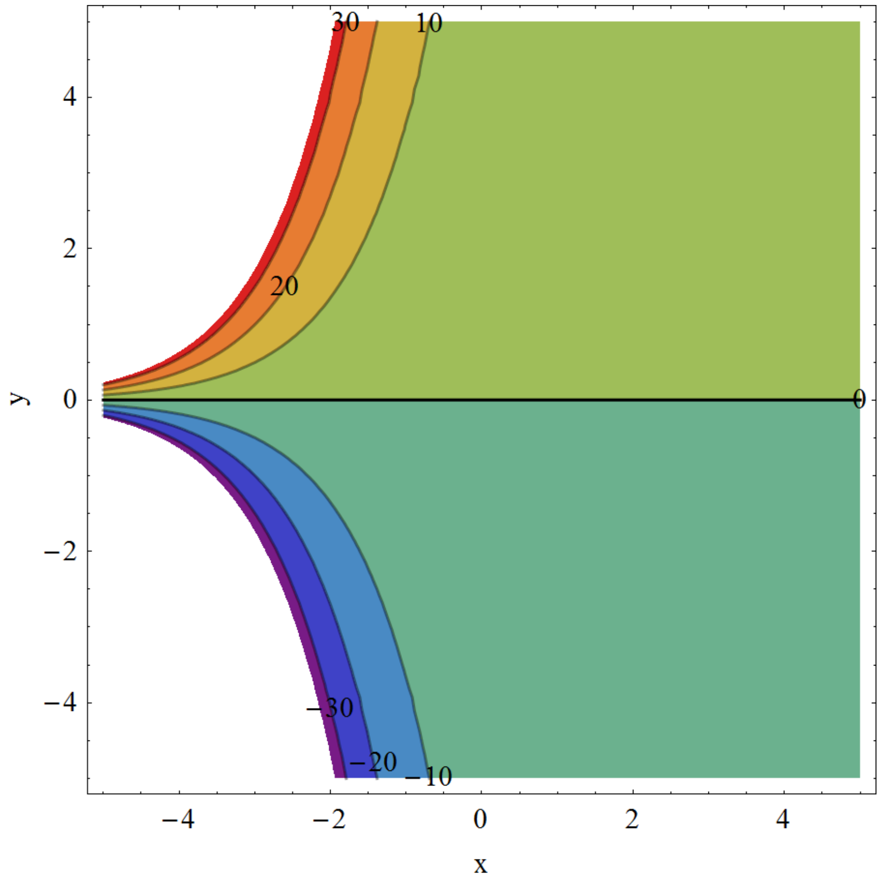

Figure 1: Plot of the characteristic curves and the data curve for $-5 < x < 5$ and $-5 < y < 5$.

Part (a)

The boundary condition is $\phi(x) = x$ when $y = 0$, so

$$u(x, 0) = f(0) = x.$$

$u = x$ doesn't satisfy the PDE. Unfortunately, a solution cannot be determined from this boundary condition.

Part (b)

The boundary condition is $\phi(x) = 1$ when $y = 0$, so

$$u(x, 0) = f(0) = 1.$$

$u = 1$ does satisfy the PDE. From this, all we can say about u is that

$$u(x, y) = \begin{cases} 1 & \text{when } y = 0 \\ f\left(ye^{-x}\right) & \text{all other } x \text{ and } y \end{cases}.$$

Therefore, the solution is not unique.