Exercise 2

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Solution

The general form of a second-order PDE is

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

It is classified as parabolic, hyperbolic, or elliptic depending on whether the discriminant, $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$, is equal to, greater than, or less than 0, respectively. In other words,

$$\text{the PDE is} \begin{cases} \text{hyperbolic} & \text{if } a_{12}^2 - a_{11}a_{22} > 0 \\ \text{parabolic} & \text{if } a_{12}^2 - a_{11}a_{22} = 0 \\ \text{elliptic} & \text{if } a_{12}^2 - a_{11}a_{22} < 0 \end{cases}.$$

For this equation the relevant coefficients are

$$\begin{aligned} a_{11} &= 1 + x \\ a_{12} &= xy \\ a_{22} &= -y^2. \end{aligned}$$

Hence, the discriminant is

$$\mathscr{D} = a_{12}^2 - a_{11}a_{22} = (xy)^2 - (1+x)(-y^2) = y^2(x^2 + x + 1).$$

$x^2 + x + 1$ is a parabola whose lowest value, $+3/4$, occurs at the vertex, $x = -1/2$. That is, it is positive for all x. y^2 is 0 when $y = 0$ and positive for all other y. Therefore, the PDE is hyperbolic for all x and y except on the line, $y = 0$, where it is parabolic.

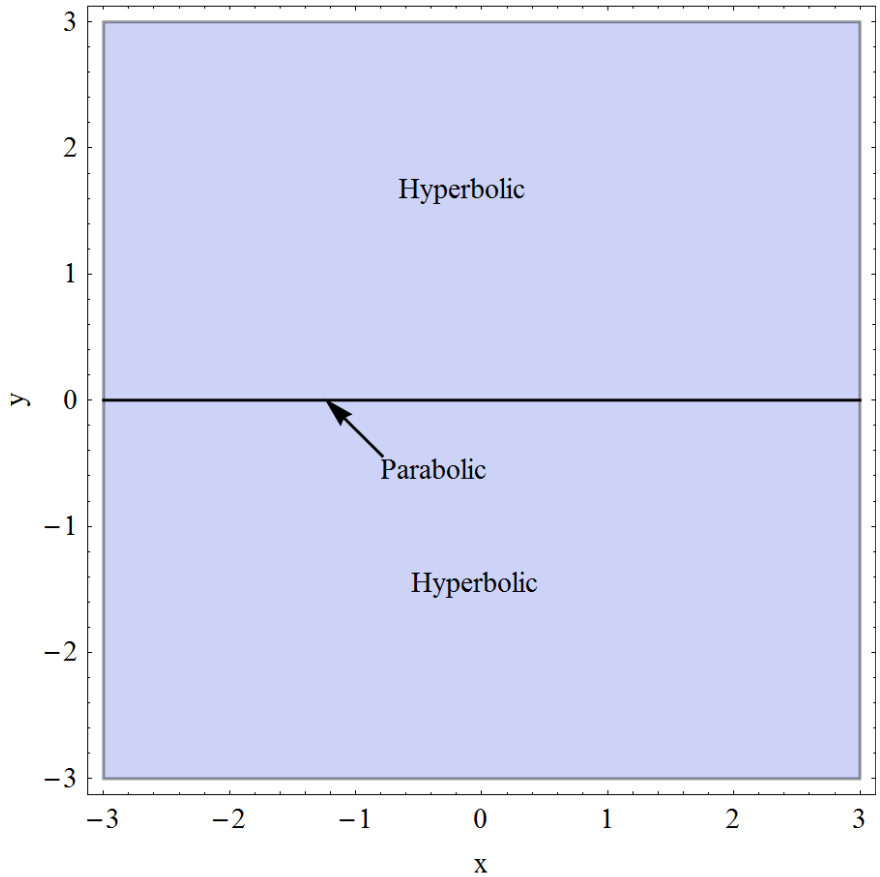

Figure 1: The different regions of the xy-plane shown for $-3 < x < 3$ and $-3 < y < 3$.