# Exercise 3

Among all the equations of the form (1), show that the only ones that are unchanged under all rotations (*rotationally invariant*) have the form $a(u_{xx} + u_{yy}) + bu = 0$.

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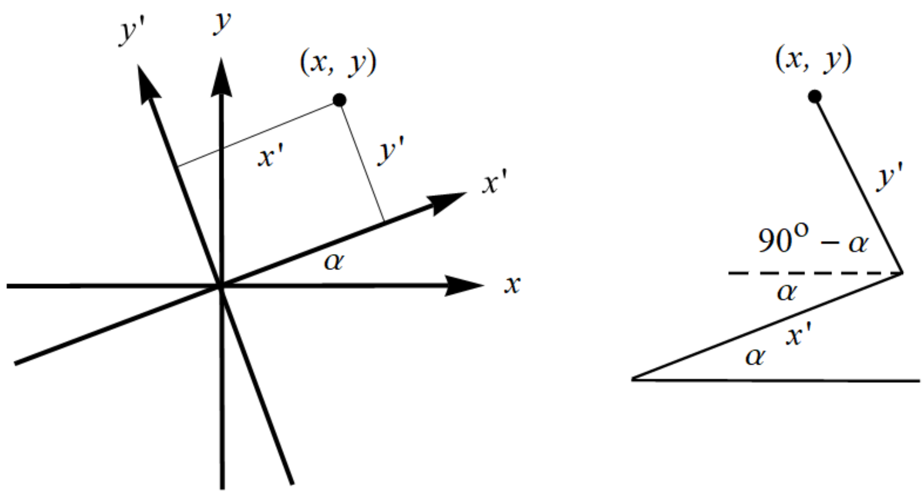
**Solution**[1]

Equation (1) in the textbook gives the general form of a linear homogeneous second-order PDE.

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0 \tag{1}$$

Suppose we define a new pair of axes in the $xy$-plane by making a rotation of angle $\alpha$.




Use this figure to write $x$ and $y$ in terms of $x'$ and $y'$.

$$x = x'\cos\alpha - y'\cos(90^\circ - \alpha)$$
$$y = x'\sin\alpha + y'\sin(90^\circ - \alpha)$$

Simplify each right side.

$$x = x'\cos\alpha - y'\sin\alpha$$
$$y = x'\sin\alpha + y'\cos\alpha$$

Solve this system of equations for $x'$ and $y'$.

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = -x\sin\alpha + y\cos\alpha$$

These equations give the coordinates of $(x, y)$ in the new, tilted coordinate system. The aim now is to make this change of variables in the PDE.

[1]Special thanks to L. Baker for pointing out my mistake.

Use the chain rule to express each of the derivatives in terms of the new variables.

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x}\\ &= u_{x'}(\cos\alpha) + u_{y'}(-\sin\alpha)\\ &= u_{x'}\cos\alpha - u_{y'}\sin\alpha\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y}\\ &= u_{x'}(\sin\alpha) + u_{y'}(\cos\alpha)\\ &= u_{x'}\sin\alpha + u_{y'}\cos\alpha\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\\ &= \left(\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'}\right)\left(\frac{\partial u}{\partial x}\right)\\ &= \left(\cos\alpha\frac{\partial}{\partial x'} - \sin\alpha\frac{\partial}{\partial y'}\right)(u_{x'}\cos\alpha - u_{y'}\sin\alpha)\\ &= \cos\alpha\frac{\partial}{\partial x'}(u_{x'}\cos\alpha - u_{y'}\sin\alpha) - \sin\alpha\frac{\partial}{\partial y'}(u_{x'}\cos\alpha - u_{y'}\sin\alpha)\\ &= \cos\alpha(u_{x'x'}\cos\alpha - u_{x'y'}\sin\alpha) - \sin\alpha(u_{y'x'}\cos\alpha - u_{y'y'}\sin\alpha)\\ &= u_{x'x'}\cos^2\alpha - 2u_{x'y'}\sin\alpha\cos\alpha + u_{y'y'}\sin^2\alpha\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\\ &= \left(\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'}\right)\left(\frac{\partial u}{\partial y}\right)\\ &= \left(\cos\alpha\frac{\partial}{\partial x'} - \sin\alpha\frac{\partial}{\partial y'}\right)(u_{x'}\sin\alpha + u_{y'}\cos\alpha)\\ &= \cos\alpha\frac{\partial}{\partial x'}(u_{x'}\sin\alpha + u_{y'}\cos\alpha) - \sin\alpha\frac{\partial}{\partial y'}(u_{x'}\sin\alpha + u_{y'}\cos\alpha)\\ &= \cos\alpha(u_{x'x'}\sin\alpha + u_{x'y'}\cos\alpha) - \sin\alpha(u_{y'x'}\sin\alpha + u_{y'y'}\cos\alpha)\\ &= u_{x'x'}\sin\alpha\cos\alpha + u_{x'y'}(\cos^2\alpha - \sin^2\alpha) - u_{y'y'}\sin\alpha\cos\alpha\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\\ &= \left(\frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'}\right)\left(\frac{\partial u}{\partial y}\right)\\ &= \left(\sin\alpha\frac{\partial}{\partial x'} + \cos\alpha\frac{\partial}{\partial y'}\right)(u_{x'}\sin\alpha + u_{y'}\cos\alpha)\\ &= \sin\alpha\frac{\partial}{\partial x'}(u_{x'}\sin\alpha + u_{y'}\cos\alpha) + \cos\alpha\frac{\partial}{\partial y'}(u_{x'}\sin\alpha + u_{y'}\cos\alpha)\\ &= \sin\alpha(u_{x'x'}\sin\alpha + u_{x'y'}\cos\alpha) + \cos\alpha(u_{y'x'}\sin\alpha + u_{y'y'}\cos\alpha)\\ &= u_{x'x'}\sin^2\alpha + 2u_{x'y'}\sin\alpha\cos\alpha + u_{y'y'}\cos^2\alpha\end{aligned}$$

Substitute each of these formulas into equation (1).

$$\begin{aligned}a_{11}(u_{x'x'}\cos^2\alpha - 2u_{x'y'}\sin\alpha\cos\alpha + u_{y'y'}\sin^2\alpha)\\ + 2a_{12}[u_{x'x'}\sin\alpha\cos\alpha + u_{x'y'}(\cos^2\alpha - \sin^2\alpha) - u_{y'y'}\sin\alpha\cos\alpha]\\ + a_{22}(u_{x'x'}\sin^2\alpha + 2u_{x'y'}\sin\alpha\cos\alpha + u_{y'y'}\cos^2\alpha)\\ + a_1(u_{x'}\cos\alpha - u_{y'}\sin\alpha) + a_2(u_{x'}\sin\alpha + u_{y'}\cos\alpha) + a_0 u = 0\end{aligned}$$

Factor each of the terms on the left side.

$$\begin{aligned}(a_{11}\cos^2\alpha + 2a_{12}\sin\alpha\cos\alpha + a_{22}\sin^2\alpha)u_{x'x'}\\ + 2[-a_{11}\sin\alpha\cos\alpha + a_{12}(\cos^2\alpha - \sin^2\alpha) + a_{22}\sin\alpha\cos\alpha]u_{x'y'}\\ + (a_{11}\sin^2\alpha - 2a_{12}\sin\alpha\cos\alpha + a_{22}\cos^2\alpha)u_{y'y'}\\ + (a_1\cos\alpha + a_2\sin\alpha)u_{x'} + (-a_1\sin\alpha + a_2\cos\alpha)u_{y'} + a_0 u = 0\end{aligned}$$

For the PDE in equation (1) to be rotationally invariant, all of the coefficients in this transformed PDE must be the same as those in the original for any value of $\alpha$.

$$\begin{aligned}a_{11} &= a_{11}\cos^2\alpha + 2a_{12}\sin\alpha\cos\alpha + a_{22}\sin^2\alpha\\ a_{12} &= -a_{11}\sin\alpha\cos\alpha + a_{12}(\cos^2\alpha - \sin^2\alpha) + a_{22}\sin\alpha\cos\alpha\\ a_{22} &= a_{11}\sin^2\alpha - 2a_{12}\sin\alpha\cos\alpha + a_{22}\cos^2\alpha\\ a_1 &= a_1\cos\alpha + a_2\sin\alpha\\ a_2 &= -a_1\sin\alpha + a_2\cos\alpha\\ a_0 &= a_0\end{aligned}$$

The sixth equation is automatically satisfied, so $a_0$ remains arbitrary.

$$a_0 = b$$

Choose a value for $\alpha$, say $\alpha = \pi/6$, and solve the resulting system of equations for $a_{11}$, $a_{12}$, $a_{22}$, $a_1$, and $a_2$.

$$\begin{aligned}a_{11} &= a_{11}\left(\frac{\sqrt{3}}{2}\right)^2 + 2a_{12}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + a_{22}\left(\frac{1}{2}\right)^2\\ a_{12} &= -a_{11}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + a_{12}\left[\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right] + a_{22}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ a_{22} &= a_{11}\left(\frac{1}{2}\right)^2 - 2a_{12}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + a_{22}\left(\frac{\sqrt{3}}{2}\right)^2\\ a_1 &= a_1\left(\frac{\sqrt{3}}{2}\right) + a_2\left(\frac{1}{2}\right)\\ a_2 &= -a_1\left(\frac{1}{2}\right) + a_2\left(\frac{\sqrt{3}}{2}\right)\end{aligned}$$

Doing so results in $a_{11} = a_{22} = a$, $a_{12} = 0$, $a_1 = 0$, and $a_2 = 0$. Therefore, among all the equations of the form (1), the only ones that are rotationally invariant have the form $a(u_{xx} + u_{yy}) + bu = 0$.