Exercise 4

What is the *type* of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g.

Solution

The general form of a second-order PDE is

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

It is classified as parabolic, hyperbolic, or elliptic depending on whether the discriminant, $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$, is equal to, greater than, or less than 0, respectively. In other words,

$$\text{the PDE is} \begin{cases} \text{hyperbolic} & \text{if } a_{12}^2 - a_{11}a_{22} > 0 \\ \text{parabolic} & \text{if } a_{12}^2 - a_{11}a_{22} = 0 \\ \text{elliptic} & \text{if } a_{12}^2 - a_{11}a_{22} < 0 \end{cases}.$$

The relevant coefficients are $a_{11} = 1$, $a_{12} = -2$, and $a_{22} = 4$. So

$$\mathscr{D} = a_{12}^2 - a_{11}a_{22} = (-2)^2 - 1 \cdot 4 = 0.$$

Therefore, the PDE is parabolic. We can verify that

$$u(x, y) = f(y + 2x) + xg(y + 2x)$$

is the correct solution.

$$\begin{aligned} u_x &= 2f' + g + 2xg' \\ u_{xx} &= 2f'' \cdot 2 + 2g' + 2g' + 4xg'' \\ u_{xy} &= 2f'' + g' + 2xg'' \\ u_y &= f' + xg' \\ u_{yy} &= f'' + xg'' \end{aligned}$$

So

$$u_{xx} - 4u_{xy} + 4u_{yy} = \cancel{4f''} + \cancel{4g'} + \cancel{4xyg''} - \cancel{8f''} - \cancel{4g'} - \cancel{8xg''} + \cancel{4f''} + \cancel{4xg''}.$$

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0.$$

Hence, $u(x, y)$ is the correct solution.