# Exercise 6

Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution. (*Hint:* Substitute $v = u_y$.)

(c) With the auxiliary conditions $u(x, 0) = e^{-3x}$ and $u_y(x, 0) = 0$, does a solution exist? Is it unique?

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**Solution**

## Part (a)

The general form of a second-order PDE is

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

It is classified as parabolic, hyperbolic, or elliptic depending on whether the discriminant, $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$, is equal to, greater than, or less than 0, respectively. In other words,

$$\text{the PDE is} \begin{cases} \text{hyperbolic} & \text{if } a_{12}^2 - a_{11}a_{22} > 0 \\ \text{parabolic} & \text{if } a_{12}^2 - a_{11}a_{22} = 0 \\ \text{elliptic} & \text{if } a_{12}^2 - a_{11}a_{22} < 0 \end{cases}.$$

For $3u_y + u_{xy} = 0$, the relevant coefficients are $a_{11} = 0$, $a_{12} = 1/2$, and $a_{22} = 0$. The discriminant is then

$$\mathscr{D} = a_{12}^2 - a_{11}a_{22} = \left(\frac{1}{2}\right)^2 - 0 \cdot 0 = \frac{1}{4}.$$

Therefore, the PDE is hyperbolic.

## Part (b)

$$3u_y + u_{xy} = 0 \quad \to \quad 3u_y + \frac{\partial}{\partial x}(u_y) = 0$$

Following the hint, substitute $v = u_y$ into the PDE.

$$3v + v_x = 0$$

Because this is a first-order linear differential equation, it can be solved by multiplying both sides by an integrating factor $I$.

$$I(x) = \exp\left(\int^x 3\,ds\right) = e^{3x}$$

Proceed with the multiplication.

$$3e^{3x}v + e^{3x}v_x = 0$$

The left side can be written as $\partial/\partial x(Iv)$ by the product rule.

$$\frac{\partial}{\partial x}\left(e^{3x}v\right) = 0$$

Integrate both sides partially with respect to $x$.

$$e^{3x}v = f(y)$$

Here $f$ is an arbitrary function of $y$. Divide both sides by $e^{3x}$ to solve for $v$.

$$v(x,y) = e^{-3x}f(y)$$

Now that $v$ is known, substitute $v = u_y$.

$$\frac{\partial u}{\partial y} = e^{-3x}f(y)$$

Solve for $u$ by integrating both sides partially with respect to $y$.

$$\boxed{u(x,y) = e^{-3x}F(y) + g(x)}$$

$F$ is an arbitrary function of $y$, and $g$ is an arbitrary function of $x$. We can verify that this is the general solution of the PDE.

$$\begin{aligned} u_y &= e^{-3x}F' \\ u_{xy} &= \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial x}(e^{-3x}F') = -3e^{-3x}F' \end{aligned}$$

$3u_y + u_{xy} = 3e^{-3x}F' - 3e^{-3x}F' = 0$, which means the solution for $u(x,y)$ is correct.

**Part (c)**

Apply the given boundary conditions to obtain conditions for $F$ and $g$.

$$\begin{aligned} u(x,0) = e^{-3x} \quad &\rightarrow \quad e^{-3x}F(0) + g(x) = e^{-3x} \quad \rightarrow \quad F(0) + g(x)e^{3x} = 1 \\ u_y(x,0) = 0 \quad &\rightarrow \quad e^{-3x}F'(0) = 0 \quad \rightarrow \quad F'(0) = 0 \end{aligned}$$

Two functions that satisfy these conditions are $F(y) = y^2$ and $g(x) = e^{-3x}$, so one solution to the boundary value problem is

$$\begin{aligned} u(x,y) &= e^{-3x}y^2 + e^{-3x} \\ &= e^{-3x}(y^2 + 1). \end{aligned}$$

Two other functions that satisfy these conditions are $F(y) = \cos y$ and $g(x) = 0$, so another solution to the boundary value problem is

$$u(x,y) = e^{-3x}\cos y.$$

Therefore, for the auxiliary conditions, $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, a solution exists but is not unique.