Exercise 4

Consider the eigenvalue problem $-\Delta v = \lambda v$ in the unit square $D = \{0 < x < 1,\ 0 < y < 1\}$ with the Dirichlet BC $v = 0$ on the bottom and both vertical sides, and the Robin BC $\partial v/\partial y = -v$ on the top $\{y = 1\}$.

(a) Show that all the eigenvalues are positive.

(b) Find an equation for the eigenvalues λ. Show that they can be expressed in terms of the roots of the equation $s + \tan s = 0$.

(c) Find the solutions of the last equation graphically. Find an approximate formula for the (m, n)th eigenvalue for large (m, n).

Solution

The eigenvalue problem to solve is as follows.

$$\begin{gathered} -\Delta v = \lambda v, \quad 0 < x < 1,\ 0 < y < 1 \\ v(0, y) = 0 \qquad v(x, 0) = 0 \\ v(1, y) = 0 \qquad \frac{\partial v}{\partial y}(x, 1) = -v(x, 1) \end{gathered}$$

Write out the Laplacian operator in Cartesian coordinates.

$$-(v_{xx} + v_{yy}) = \lambda v.$$

Note that this PDE is known as the Helmholtz equation. Because it and the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $v(x, y) = X(x)Y(y)$ and substitute it into the PDE

$$-(v_{xx} + v_{yy}) = \lambda v \quad \rightarrow \quad -(X''Y + XY'') = \lambda XY$$

and the boundary conditions.

$$\begin{aligned} v(0, y) &= 0 & &\rightarrow & X(0)Y(y) &= 0 & &\rightarrow & X(0) &= 0 \\ v(1, y) &= 0 & &\rightarrow & X(1)Y(y) &= 0 & &\rightarrow & X(1) &= 0 \\ v(x, 0) &= 0 & &\rightarrow & X(x)Y(0) &= 0 & &\rightarrow & Y(0) &= 0 \\ \frac{\partial v}{\partial y}(x, 1) &= -v(x, 1) & &\rightarrow & X(x)Y'(1) &= -X(x)Y(1) & &\rightarrow & Y'(1) &= -Y(1) \end{aligned}$$

Divide both sides by $-XY$ in the PDE in order to separate variables.

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

Bring Y''/Y to the right side.

$$\underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{-\lambda - \frac{Y''}{Y}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant μ.

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = \mu$$

As a result of applying the method of separation of variables, the PDE has been reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{X''}{X} &= \mu \\ -\lambda - \frac{Y''}{Y} &= \mu \end{aligned}\right\}$$

Start by solving the ODE for X.

$$X'' = \mu X$$

Suppose first that μ is positive: $\mu = \alpha^2$.

$$X'' = \alpha^2 X$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(1) &= C_1 \cosh \alpha + C_2 \sinh \alpha = 0 \end{aligned}$$

The second equation reduces to $C_2 \sinh \alpha = 0$. No nonzero value of α satisfies it, so C_2 must be zero. The trivial solution $X(x) = 0$ is obtained, which means μ is not positive. Suppose secondly that μ is zero: $\mu = 0$. The ODE for X becomes

$$X'' = 0$$

Integrate both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(1) &= C_3 + C_4 = 0 \end{aligned}$$

The second equation reduces to $C_3 = 0$. The trivial solution results, which means μ is not zero. Suppose thirdly that μ is negative: $\mu = -\beta^2$. The ODE for X becomes

$$X'' = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(1) &= C_5 \cos\beta + C_6 \sin\beta = 0 \end{aligned}$$

The second equation reduces to $C_6 \sin\beta = 0$. To avoid the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin\beta &= 0 \\ \beta_n &= n\pi, \quad n = 1, 2, \ldots. \end{aligned}$$

There are negative eigenvalues $\mu = -n^2\pi^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos\beta x + C_6 \sin\beta x \\ &= C_6 \sin\beta x \quad \rightarrow \quad X_n(x) = \sin n\pi x. \end{aligned}$$

With $\mu = -n^2\pi^2$, solve the ODE for Y.

$$-\lambda - \frac{Y''}{Y} = -n^2\pi^2$$

$$Y'' = -(\lambda - n^2\pi^2)Y$$

Suppose first that λ is positive: $\lambda = \xi^2$.

$$Y'' = -(\xi^2 - n^2\pi^2)Y$$

The general solution is written in terms of sine and cosine.

$$Y(y) = C_7 \cos\sqrt{\xi^2 - n^2\pi^2}y + C_8 \sin\sqrt{\xi^2 - n^2\pi^2}y$$

Take a derivative of it.

$$Y'(y) = \sqrt{\xi^2 - n^2\pi^2}(-C_7 \sin\sqrt{\xi^2 - n^2\pi^2}y + C_8 \cos\sqrt{\xi^2 - n^2\pi^2}y)$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} Y(0) &= C_7 = 0 \\ Y'(1) + Y(1) &= \sqrt{\xi^2 - n^2\pi^2}(-C_7 \sin\sqrt{\xi^2 - n^2\pi^2} + C_8 \cos\sqrt{\xi^2 - n^2\pi^2}) \\ &\quad + C_7 \cos\sqrt{\xi^2 - n^2\pi^2} + C_8 \sin\sqrt{\xi^2 - n^2\pi^2} = 0 \end{aligned}$$

The second equation reduces to $C_8(\sqrt{\xi^2 - n^2\pi^2} \cos\sqrt{\xi^2 - n^2\pi^2} + \sin\sqrt{\xi^2 - n^2\pi^2}) = 0$. To avoid getting the trivial solution, we insist that $C_8 \neq 0$. Then

$$\begin{aligned} \sqrt{\xi^2 - n^2\pi^2} \cos\sqrt{\xi^2 - n^2\pi^2} + \sin\sqrt{\xi^2 - n^2\pi^2} &= 0 \\ \sqrt{\xi^2 - n^2\pi^2} + \tan\sqrt{\xi^2 - n^2\pi^2} &= 0. \end{aligned}$$

Letting $s = \sqrt{\xi^2 - n^2\pi^2}$, the equation for the positive eigenvalues is thus $\tan s + s = 0$.

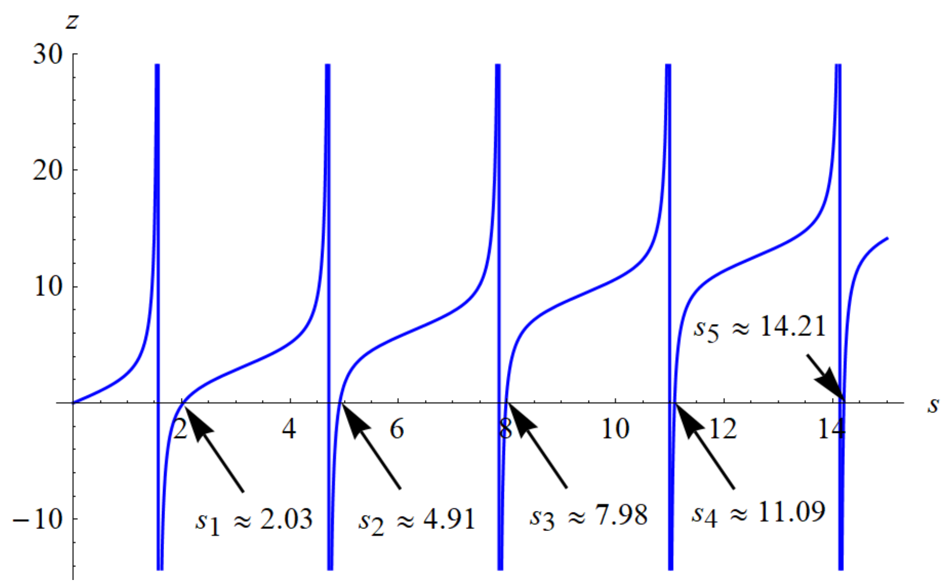

Figure 1: This figure shows a plot of $z = \tan s + s$ versus s. Each of the zeros corresponds to a positive eigenvalue $\lambda = s^2 + n^2\pi^2$. The first five values for s are shown in the graph; they are approximately $s_1 \approx 2.03$, $s_2 \approx 4.91$, $s_3 \approx 7.98$, $s_4 \approx 11.09$, and $s_5 \approx 14.21$. Notice that if s is high enough, the zeros can be approximated by the vertical asymptotes of tangent, $s = \frac{\pi}{2} + m\pi$, where $m = 1, 2, \ldots$. For large (m, n), therefore, the positive eigenvalues are given by $\lambda \sim \left(\frac{\pi}{2} + m\pi\right)^2 + n^2\pi^2$.

The eigenfunctions associated with them are

$$\begin{aligned} Y(y) &= C_7 \cos\sqrt{\xi^2 - n^2\pi^2}y + C_8 \sin\sqrt{\xi^2 - n^2\pi^2}y \\ &= C_8 \sin\sqrt{\xi^2 - n^2\pi^2}y \quad \to \quad Y_m(y) = \sin s_m y. \end{aligned}$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for Y becomes

$$Y'' = n^2\pi^2 Y.$$

The general solution is in terms of hyperbolic sine and hyperbolic cosine, which leads to the trivial solution as before. Consequently, zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\zeta^2$. The ODE for Y becomes

$$Y'' = (\zeta^2 + n^2\pi^2)Y.$$

Again, the general solution is in terms of hyperbolic sine and hyperbolic cosine, which leads to the trivial solution. Consequently, there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for v is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$v(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin n\pi x \sin s_m y$$

s_m is the mth solution to $\tan s + s = 0$, and $\lambda = s_m^2 + n^2\pi^2$.