Exercise 7

Prove Theorem 2 in the Neumann and Robin cases.

Solution

Green's first identity states that for any two functions, u and v,

$$\iint_{\text{bdy } D} v\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla v\cdot\nabla u\,d\mathbf{x} + \iiint_D v\Delta u\,d\mathbf{x}.$$

Let w satisfy the Helmholtz equation in some domain D

$$-\Delta w = \lambda w \quad \text{in } D$$

subject to a homogeneous Dirichlet, Neumann, or Robin boundary condition on the boundary of D.

<u>**Dirichlet Case**</u>

In this case the boundary condition is homogeneous Dirichlet:

$$w = 0 \quad \text{on bdy } D.$$

Let $u = w$ and $v = w$ in Green's first identity.

$$\iint_{\text{bdy } D} \overbrace{w}^{=0}\frac{\partial w}{\partial n}\,dS = \iiint_D \nabla w\cdot\nabla w\,d\mathbf{x} + \iiint_D w\Delta w\,d\mathbf{x}$$

The surface integral on the left vanishes. Substitute $-\lambda w$ for Δw on the right.

$$0 = \iiint_D |\nabla w|^2\,d\mathbf{x} + \iiint_D w(-\lambda w)\,d\mathbf{x}$$

Solve for λ.

$$\lambda = \frac{\iiint_D |\nabla w|^2\,d\mathbf{x}}{\iiint_D w^2\,d\mathbf{x}}$$

Since the integrands are positive, λ is positive as well. However, λ is not zero because that would ultimately imply that $w = 0$, the trivial solution. The eigenfunction w has to be nontrivial.

$$\lambda = 0 \quad\rightarrow\quad \iiint_D |\nabla w|^2\,d\mathbf{x} = 0 \quad\rightarrow\quad |\nabla w|^2 = 0 \quad\rightarrow\quad \nabla w = 0 \quad\rightarrow\quad w = C_1 = 0$$

The constant C_1 must be zero to be consistent with $w = 0$ on the boundary of D.

Neumann Case

In this case the boundary condition is homogeneous Neumann:

$$\frac{\partial w}{\partial n} = 0 \quad \text{on bdy } D.$$

Let $u = w$ and $v = w$ in Green's first identity.

$$\iint_{\text{bdy } D} w \overbrace{\frac{\partial w}{\partial n}}^{=0} \, dS = \iiint_D \nabla w \cdot \nabla w \, d\mathbf{x} + \iiint_D w \Delta w \, d\mathbf{x}$$

The surface integral on the left vanishes. Substitute $-\lambda w$ for Δw on the right.

$$0 = \iiint_D |\nabla w|^2 \, d\mathbf{x} + \iiint_D w(-\lambda w) \, d\mathbf{x}$$

Solve for λ.

$$\lambda = \frac{\iiint_D |\nabla w|^2 \, d\mathbf{x}}{\iiint_D w^2 \, d\mathbf{x}}$$

Since the integrands are positive, λ is positive as well. Unlike the Dirichlet case, λ may be zero.

$$\lambda = 0 \quad \rightarrow \quad \iiint_D |\nabla w|^2 \, d\mathbf{x} = 0 \quad \rightarrow \quad |\nabla w|^2 = 0 \quad \rightarrow \quad \nabla w = 0 \quad \rightarrow \quad w = C_2$$

C_2 remains arbitrary because only the derivative is specified on the boundary of D.

Robin Case

In this case the boundary condition is homogeneous Robin:

$$\frac{\partial w}{\partial n} + aw = 0 \quad \text{on bdy } D.$$

Let $u = w$ and $v = w$ in Green's first identity.

$$\iint_{\text{bdy } D} w \frac{\partial w}{\partial n} \, dS = \iiint_D \nabla w \cdot \nabla w \, d\mathbf{x} + \iiint_D w \Delta w \, d\mathbf{x}$$

Substitute $-aw$ for $\partial w/\partial n$ on the left and substitute $-\lambda w$ for Δw on the right.

$$\iint_{\text{bdy } D} w(-aw) \, dS = \iiint_D |\nabla w|^2 \, d\mathbf{x} + \iiint_D w(-\lambda w) \, d\mathbf{x}$$

Solve for λ.

$$\lambda = \frac{\iiint_D |\nabla w|^2 \, d\mathbf{x} + a \iint_{\text{bdy } D} w^2 \, dS}{\iiint_D w^2 \, d\mathbf{x}}$$

Provided that $a \geq 0$, λ is positive since each of the integrands is positive. Unlike the Dirichlet case, λ may be zero if $a = 0$.

$$\lambda = 0 \quad \rightarrow \quad \iiint_D |\nabla w|^2 \, d\mathbf{x} + a \iint_{\text{bdy } D} w^2 \, dS = 0 \quad \rightarrow \quad \iiint_D |\nabla w|^2 \, d\mathbf{x} = 0 \quad \rightarrow \quad |\nabla w|^2 = 0$$

$$\nabla w = 0$$

$$w = C_3$$

C_3 remains arbitrary because only the derivative is specified on the boundary of D.