Exercise 6

Do the same for the annulus $\{a^2 < x^2 + y^2 < b^2\}$ with $u = B$ on the whole boundary.

Solution

The initial boundary value problem to solve is

$$\begin{aligned} u_t = k\nabla^2 u &= k\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad a \le r \le b,\ 0 \le \theta \le 2\pi \\ u(a,\theta,t) &= B \\ u(b,\theta,t) &= B \\ u(r,\theta,0) &= 0. \end{aligned}$$

Since the boundary conditions are independent of θ, we assume that the solution is as well.

$$u = u(r,t)$$

The angular derivative in the PDE vanishes as a result.

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r\right)$$

To make the boundary conditions homogeneous, make the following change of variables.

$$v(r,t) = u(r,t) - B$$

Find the derivatives of u in terms of this new variable.

$$\begin{aligned} u_t &= v_t \\ u_r &= v_r \\ u_{rr} &= v_{rr} \end{aligned}$$

Consequently, v satisfies the diffusion equation as well.

$$v_t = k\left(v_{rr} + \frac{1}{r}v_r\right)$$

The conditions associated with this equation are

$$\begin{aligned} v(a,t) &= u(a,t) - B = B - B = 0 \\ v(b,t) &= u(b,t) - B = B - B = 0 \\ v(r,0) &= u(r,0) - B = 0 - B = -B. \end{aligned}$$

Because the PDE for v and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $v = R(r)T(t)$ and substitute it into the PDE

$$v_t = k\left(v_{rr} + \frac{1}{r}v_r\right) \quad \rightarrow \quad \frac{\partial}{\partial t}[R(r)T(t)] = k\left[\frac{\partial^2}{\partial r^2}[R(r)T(t)] + \frac{1}{r}\frac{\partial}{\partial r}[R(r)T(t)]\right]$$

and the boundary conditions.

$$\begin{aligned} v(a,t) = 0 &\quad\rightarrow\quad R(a)T(t) = 0 &\quad\rightarrow\quad R(a) = 0 \\ v(b,t) = 0 &\quad\rightarrow\quad R(b)T(t) = 0 &\quad\rightarrow\quad R(b) = 0 \end{aligned}$$

Proceed to separate variables in the PDE.

$$R\frac{dT}{dt} = kT\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right)$$

Divide both sides by $kR(r)T(t)$.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right)}_{\text{function of } r}$$

The only way a function of t can be equal to a function of r is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in t and one in r.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) &= \lambda \end{aligned}\right\}$$

Positive values of λ can be disregarded because the solution to the ODE for T will diverge as $t \to \infty$. On the contrary, the temperature will eventually reach a steady state since the boundaries are held at a constant temperature B. Suppose first that λ is zero: $\lambda = 0$. The ODE for R becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = 0.$$

Multiply both sides by r.

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} = 0$$

Write the left side as a derivative using the chain rule.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = C_1$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = C_1 \ln r + C_2$$

Apply the boundary conditions to determine C_1 and C_2.

$$R(a) = C_1 \ln a + C_2 = 0$$
$$R(b) = C_1 \ln b + C_2 = 0$$

Subtract both sides of the second equation from those of the first.

$$C_1(\ln a - \ln b) = 0 \quad \rightarrow \quad C_1 = 0$$

Add both sides of the second equation to those of the first.

$$C_1(\ln a + \ln b) + 2C_2 = 0 \quad \rightarrow \quad C_2 = 0$$

The trivial solution $R(r) = 0$ is obtained, so zero is not an eigenvalue. Suppose secondly that λ is negative: $\lambda = -\gamma^2$. The ODE for R becomes

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + \gamma^2 r^2 R = 0$$

The general solution is written in terms of J_0 and Y_0, the zero-order Bessel functions of the first and second kind, respectively.

$$R(r) = C_3 J_0(\gamma r) + C_4 Y_0(\gamma r)$$

Apply the boundary conditions.

$$R(a) = C_3 J_0(\gamma a) + C_4 Y_0(\gamma a) = 0$$
$$R(b) = C_3 J_0(\gamma b) + C_4 Y_0(\gamma b) = 0$$

Solve the first equation for C_3

$$C_3 = -C_4 \frac{Y_0(\gamma a)}{J_0(\gamma a)}$$

and substitute it into the second equation.

$$-C_4 \frac{Y_0(\gamma a)}{J_0(\gamma a)} J_0(\gamma b) + C_4 Y_0(\gamma b) = 0$$

Multiply both sides by $J_0(\gamma a)/C_4$.

$$J_0(\gamma a) Y_0(\gamma b) - J_0(\gamma b) Y_0(\gamma a) = 0$$

The negative eigenvalues are thus $\lambda = -\gamma_n^2$ $(n = 1, 2, \ldots)$, where γ_n satisfies

$$J_0(\gamma_n a) Y_0(\gamma_n b) - J_0(\gamma_n b) Y_0(\gamma_n a) = 0,$$

and the eigenfunctions associated with them are

$$\begin{aligned} R(r) &= C_3 J_0(\gamma r) + C_4 Y_0(\gamma r) \\ &= -C_4 \frac{Y_0(\gamma a)}{J_0(\gamma a)} J_0(\gamma r) + C_4 Y_0(\gamma r) \\ &= \frac{C_4}{J_0(\gamma a)} [J_0(\gamma a) Y_0(\gamma r) - J_0(\gamma r) Y_0(\gamma a)] \\ &= C_5 [J_0(\gamma a) Y_0(\gamma r) - J_0(\gamma r) Y_0(\gamma a)] \quad \rightarrow \quad R_n(r) = J_0(\gamma_n a) Y_0(\gamma_n r) - J_0(\gamma_n r) Y_0(\gamma_n a). \end{aligned}$$

With $\lambda = -\gamma_n^2$, solve the ODE for T now.

$$\frac{dT}{dt} = -k\gamma_n^2 T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_6 e^{-k\gamma_n^2 t}$$

According to the principle of superposition, the general solution to the PDE for v is a linear combination of the eigenfunctions $R(r)T(t)$ over all the eigenvalues.

$$v(r,t) = \sum_{n=1}^{\infty} A_n e^{-k\gamma_n^2 t}[J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)]$$

Use the initial condition $v(r,0) = -B$ now to determine the coefficients A_n.

$$v(r,0) = \sum_{n=1}^{\infty} A_n[J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)] = -B$$

Multiply both sides by $[J_0(\gamma_m a)Y_0(\gamma_m r) - J_0(\gamma_m r)Y_0(\gamma_m a)]r$, where m is a positive integer,

$$\sum_{n=1}^{\infty} A_n[J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)][J_0(\gamma_m a)Y_0(\gamma_m r) - J_0(\gamma_m r)Y_0(\gamma_m a)]r$$
$$= -B[J_0(\gamma_m a)Y_0(\gamma_m r) - J_0(\gamma_m r)Y_0(\gamma_m a)]r$$

and then integrate both sides with respect to r from a to b.

$$\int_a^b \sum_{n=1}^{\infty} A_n[J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)][J_0(\gamma_m a)Y_0(\gamma_m r) - J_0(\gamma_m r)Y_0(\gamma_m a)]r\,dr$$
$$= \int_a^b (-B)[J_0(\gamma_m a)Y_0(\gamma_m r) - J_0(\gamma_m r)Y_0(\gamma_m a)]r\,dr$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} A_n \int_a^b [J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)][J_0(\gamma_m a)Y_0(\gamma_m r) - J_0(\gamma_m r)Y_0(\gamma_m a)]r\,dr$$
$$= -B\int_a^b [J_0(\gamma_m a)Y_0(\gamma_m r) - J_0(\gamma_m r)Y_0(\gamma_m a)]r\,dr$$

Because the Bessel functions are orthogonal with respect to the weight r, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$A_n \int_a^b [J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)]^2 r\,dr = -B\int_a^b [J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)]r\,dr$$

Solve for A_n.

$$A_n = -B\frac{\int_a^b [J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)]r\,dr}{\int_a^b [J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)]^2 r\,dr}$$

Evaluating the integrals and fully simplifying yields

$$A_n = -2\pi B\frac{2 + \pi b\gamma_n[J_0(\gamma_n a)Y_1(\gamma_n b) - J_1(\gamma_n b)Y_0(\gamma_n a)]}{-4 + \pi^2 b^2\gamma_n^2[J_0(\gamma_n a)Y_1(\gamma_n b) - J_1(\gamma_n b)Y_0(\gamma_n a)]^2}.$$

So then

$$v(r,t) = \sum_{n=1}^{\infty}(-2\pi B)\frac{2 + \pi b\gamma_n[J_0(\gamma_n a)Y_1(\gamma_n b) - J_1(\gamma_n b)Y_0(\gamma_n a)]}{-4 + \pi^2 b^2\gamma_n^2[J_0(\gamma_n a)Y_1(\gamma_n b) - J_1(\gamma_n b)Y_0(\gamma_n a)]^2}e^{-k\gamma_n^2 t}[J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)].$$

Therefore, since $u(r,t) = B + v(r,t)$,

$$u(r,t) = B - 2\pi B\sum_{n=1}^{\infty}\frac{2 + \pi b\gamma_n[J_0(\gamma_n a)Y_1(\gamma_n b) - J_1(\gamma_n b)Y_0(\gamma_n a)]}{-4 + \pi^2 b^2\gamma_n^2[J_0(\gamma_n a)Y_1(\gamma_n b) - J_1(\gamma_n b)Y_0(\gamma_n a)]^2}e^{-k\gamma_n^2 t}[J_0(\gamma_n a)Y_0(\gamma_n r) - J_0(\gamma_n r)Y_0(\gamma_n a)],$$

where γ_n satisfies

$$J_0(\gamma_n a)Y_0(\gamma_n b) - J_0(\gamma_n b)Y_0(\gamma_n a) = 0.$$