Exercise 1

Show that with the initial conditions (26), all the $\cos\sqrt{\lambda}ct$ terms in the series (18) are missing. Also show that $D_{nm} = C_{nm} = 0$ for $n \neq 0$.

[TYPO: The initial conditions are in equation (25) of the text.]

Solution

The wave equation with a Dirichlet boundary condition governs the vibrations of a membrane with a fixed boundary. Let it be subject to the initial conditions in equation (25).

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi \\ u(a,\theta,t) &= 0 \\ u(r,\theta,0) &= 0 \\ u_t(r,\theta,0) &= \psi(r) \end{aligned}$$

Equation (18) gives the general solution to this problem.

$$\begin{aligned} u(r,\theta,t) = \sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r)(A_{0m}\cos\sqrt{\lambda_{0m}}ct + C_{0m}\sin\sqrt{\lambda_{0m}}ct) \\ + \sum_{m,n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\left[(A_{nm}\cos n\theta + B_{nm}\sin n\theta)\cos\sqrt{\lambda_{nm}}ct\right. \\ \left. +(C_{nm}\cos n\theta + D_{nm}\sin n\theta)\sin\sqrt{\lambda_{nm}}ct\right] \end{aligned} \tag{18}$$

Take a derivative of it with respect to t.

$$\begin{aligned} u_t(r,\theta,t) = \sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r)\sqrt{\lambda_{0m}}c(-A_{0m}\sin\sqrt{\lambda_{0m}}ct + C_{0m}\cos\sqrt{\lambda_{0m}}ct) \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\left[-\sqrt{\lambda_{nm}}c(A_{nm}\cos n\theta + B_{nm}\sin n\theta)\sin\sqrt{\lambda_{nm}}ct\right. \\ \left. +\sqrt{\lambda_{nm}}c(C_{nm}\cos n\theta + D_{nm}\sin n\theta)\cos\sqrt{\lambda_{nm}}ct\right] \end{aligned}$$

Apply the initial conditions to determine the coefficients.

$$u(r,\theta,0) = \sum_{m=1}^{\infty} A_{0m}J_0(\sqrt{\lambda_{0m}}r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)(A_{nm}\cos n\theta + B_{nm}\sin n\theta) = 0 \tag{1}$$

$$u_t(r,\theta,0) = \sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r)\sqrt{\lambda_{0m}}cC_{0m} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}c(C_{nm}\cos n\theta + D_{nm}\sin n\theta) = \phi(r) \tag{2}$$

To find A_{0m}, integrate both sides of equation (1) with respect to θ from 0 to 2π.

$$\int_0^{2\pi}\left[\sum_{m=1}^{\infty} A_{0m}J_0(\sqrt{\lambda_{0m}}r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)(A_{nm}\cos n\theta + B_{nm}\sin n\theta)\right]d\theta = 0$$

Split up the integral on the left and bring the constants in front.

$$\sum_{m=1}^{\infty} A_{0m} J_0(\sqrt{\lambda_{0m}} r) \int_0^{2\pi} d\theta + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}} r) \left(A_{nm} \underbrace{\int_0^{2\pi} \cos n\theta \, d\theta}_{=0} + B_{nm} \underbrace{\int_0^{2\pi} \sin n\theta \, d\theta}_{=0} \right) = 0$$

Evaluate the integrals.

$$\sum_{m=1}^{\infty} A_{0m} J_0(\sqrt{\lambda_{0m}} r)(2\pi) = 0$$

Multiply both sides by $J_0(\sqrt{\lambda_{0q}} r) r$, where q is a positive integer.

$$2\pi \sum_{m=1}^{\infty} A_{0m} J_0(\sqrt{\lambda_{0m}} r) J_0(\sqrt{\lambda_{0q}} r) r = 0$$

Integrate both sides with respect to r from 0 to a.

$$\int_0^a 2\pi \sum_{m=1}^{\infty} A_{0m} J_0(\sqrt{\lambda_{0m}} r) J_0(\sqrt{\lambda_{0q}} r) r \, dr = 0$$

Bring the constants in front.

$$2\pi \sum_{m=1}^{\infty} A_{0m} \int_0^a J_0(\sqrt{\lambda_{0m}} r) J_0(\sqrt{\lambda_{0q}} r) r \, dr = 0$$

Because the Bessel functions are orthogonal with respect to the weight r, the integral is zero if $m \neq q$. As a result, every term in the infinite series vanishes except for the $m = q$ one.

$$2\pi A_{0m} \int_0^a J_0^2(\sqrt{\lambda_{0m}} r) r \, dr = 0$$

This integral is known.

$$2\pi A_{0m} \left[\frac{a^2}{2} J_1^2(\lambda_{0m} a) \right] = 0$$

So then

$$\boxed{A_{0m} = 0.}$$

To find A_{nm}, start by multiplying both sides of equation (1) by $\cos p\theta$, where p is a positive integer.

$$\sum_{m=1}^{\infty} A_{0m} J_0(\sqrt{\lambda_{0m}} r) \cos p\theta + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}} r)(A_{nm} \cos n\theta \cos p\theta + B_{nm} \sin n\theta \cos p\theta) = 0$$

Integrate both sides with respect to θ from 0 to 2π.

$$\int_0^{2\pi} \left[\sum_{m=1}^{\infty} A_{0m} J_0(\sqrt{\lambda_{0m}} r) \cos p\theta + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}} r)(A_{nm} \cos n\theta \cos p\theta + B_{nm} \sin n\theta \cos p\theta) \right] d\theta = 0$$

Split up the integral on the left and bring the constants in front.

$$\sum_{m=1}^{\infty} A_{0m} J_0(\sqrt{\lambda_{0m}} r) \underbrace{\int_0^{2\pi} \cos p\theta \, d\theta}_{=0} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}} r) \left(A_{nm} \int_0^{2\pi} \cos n\theta \cos p\theta \, d\theta + B_{nm} \underbrace{\int_0^{2\pi} \sin n\theta \cos p\theta \, d\theta}_{=0} \right) = 0$$

Evaluate the integrals.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm}J_n(\sqrt{\lambda_{nm}}r)\int_0^{2\pi}\cos n\theta\cos p\theta\,d\theta = 0$$

Because the cosine functions are orthogonal, the remaining integral is zero if $n \neq p$. As a result, every term in the infinite series over n vanishes except for the $n = p$ one.

$$\sum_{m=1}^{\infty} A_{nm}J_n(\sqrt{\lambda_{nm}}r)\int_0^{2\pi}\cos^2 n\theta\,d\theta = 0$$

Evaluate the integral.

$$\sum_{m=1}^{\infty} A_{nm}J_n(\sqrt{\lambda_{nm}}r)(\pi) = 0$$

Multiply both sides by $J_n(\sqrt{\lambda_{nq}}r)r$, where q is a positive integer.

$$\pi\sum_{m=1}^{\infty} A_{nm}J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nq}}r)r = 0$$

Integrate both sides with respect to r from 0 to a.

$$\int_0^a \pi\sum_{m=1}^{\infty} A_{nm}J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nq}}r)r\,dr = 0$$

Bring the constants in front of the integral.

$$\pi\sum_{m=1}^{\infty} A_{nm}\int_0^a J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nq}}r)r\,dr = 0$$

Because the Bessel functions are orthogonal with respect to the weight r, the integral is zero if $m \neq q$. As a result, every term in the infinite series vanishes except for the $m = q$ one.

$$\pi A_{nm}\int_0^a J_n^2(\sqrt{\lambda_{nm}}r)r\,dr = 0$$

Evaluate the integral.

$$\pi A_{nm}\left[\frac{a^2}{2}J_{n+1}^2(\sqrt{\lambda_{nm}}a)\right] = 0$$

So then

$$\boxed{A_{nm} = 0.}$$

To find B_{nm}, start by multiplying both sides of equation (1) by $\sin p\theta$, where p is a positive integer.

$$\sum_{m=1}^{\infty} A_{0m}J_0(\sqrt{\lambda_{0m}}r)\sin p\theta + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)(A_{nm}\cos n\theta\sin p\theta + B_{nm}\sin n\theta\sin p\theta) = 0$$

Integrate both sides with respect to θ from 0 to 2π.

$$\int_0^{2\pi}\left[\sum_{m=1}^{\infty} A_{0m}J_0(\sqrt{\lambda_{0m}}r)\sin p\theta + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)(A_{nm}\cos n\theta\sin p\theta + B_{nm}\sin n\theta\sin p\theta)\right]d\theta = 0$$

Split up the integral on the left and bring the constants in front.

$$\sum_{m=1}^{\infty} A_{0m} J_0(\sqrt{\lambda_{0m}}r) \underbrace{\int_0^{2\pi} \sin p\theta \, d\theta}_{=0} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\left(A_{nm} \underbrace{\int_0^{2\pi} \cos n\theta \sin p\theta \, d\theta}_{=0} + B_{nm} \int_0^{2\pi} \sin n\theta \sin p\theta \, d\theta\right) = 0$$

Evaluate the integrals.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{nm} J_n(\sqrt{\lambda_{nm}}r) \int_0^{2\pi} \sin n\theta \sin p\theta \, d\theta = 0$$

Because the sine functions are orthogonal, the remaining integral is zero if $n \neq p$. As a result, every term in the infinite series over n vanishes except for the $n = p$ one.

$$\sum_{m=1}^{\infty} B_{nm} J_n(\sqrt{\lambda_{nm}}r) \int_0^{2\pi} \sin^2 n\theta \, d\theta = 0$$

Evaluate the integral.

$$\sum_{m=1}^{\infty} B_{nm} J_n(\sqrt{\lambda_{nm}}r)(\pi) = 0$$

Multiply both sides by $J_n(\sqrt{\lambda_{nq}}r)r$, where q is a positive integer.

$$\pi \sum_{m=1}^{\infty} B_{nm} J_n(\sqrt{\lambda_{nm}}r) J_n(\sqrt{\lambda_{nq}}r) r = 0$$

Integrate both sides with respect to r from 0 to a.

$$\int_0^a \pi \sum_{m=1}^{\infty} B_{nm} J_n(\sqrt{\lambda_{nm}}r) J_n(\sqrt{\lambda_{nq}}r) r \, dr = 0$$

Bring the constants in front of the integral.

$$\pi \sum_{m=1}^{\infty} B_{nm} \int_0^a J_n(\sqrt{\lambda_{nm}}r) J_n(\sqrt{\lambda_{nq}}r) r \, dr = 0$$

Because the Bessel functions are orthogonal with respect to the weight r, the integral is zero if $m \neq q$. As a result, every term in the infinite series vanishes except for the $m = q$ one.

$$\pi B_{nm} \int_0^a J_n^2(\sqrt{\lambda_{nm}}r) r \, dr = 0$$

Evaluate the integral.

$$\pi B_{nm} \left[\frac{a^2}{2} J_{n+1}^2(\sqrt{\lambda_{nm}}a)\right] = 0$$

So then

$$\boxed{B_{nm} = 0.}$$

Therefore, all the $\cos\sqrt{\lambda}ct$ terms in the series (18) are missing. To find C_{nm}, start by multiplying both sides of equation (2) by $\cos p\theta$, where p is a positive integer.

$$\sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r)\sqrt{\lambda_{0m}}cC_{0m}\cos p\theta + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}c(C_{nm}\cos n\theta\cos p\theta + D_{nm}\sin n\theta\cos p\theta) = \phi(r)\cos p\theta$$

Integrate both sides with respect to θ from 0 to 2π.

$$\int_0^{2\pi}\left[\sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r)\sqrt{\lambda_{0m}}cC_{0m}\cos p\theta + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}c(C_{nm}\cos n\theta\cos p\theta + D_{nm}\sin n\theta\cos p\theta)\right]d\theta = \int_0^{2\pi}\phi(r)\cos p\theta\,d\theta$$

Split up the integrals and bring the constants in front.

$$\sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r)\sqrt{\lambda_{0m}}cC_{0m}\underbrace{\int_0^{2\pi}\cos p\theta\,d\theta}_{=0} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}c\left(C_{nm}\int_0^{2\pi}\cos n\theta\cos p\theta\,d\theta + D_{nm}\underbrace{\int_0^{2\pi}\sin n\theta\cos p\theta\,d\theta}_{=0}\right)d\theta = \phi(r)\underbrace{\int_0^{2\pi}\cos p\theta\,d\theta}_{=0}$$

Evaluate the integrals.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}cC_{nm}\int_0^{2\pi}\cos n\theta\cos p\theta\,d\theta = 0$$

Because the cosine functions are orthogonal, the remaining integral is zero if $n \neq p$. As a result, every term in the infinite series over n vanishes except for the $n = p$ one.

$$\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}cC_{nm}\int_0^{2\pi}\cos^2 n\theta\,d\theta = 0$$

Evaluate the integral.

$$\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}cC_{nm}(\pi) = 0$$

Multiply both sides by $J_n(\sqrt{\lambda_{nq}}r)r$, where q is a positive integer.

$$c\pi\sum_{m=1}^{\infty}\sqrt{\lambda_{nm}}C_{nm}J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nq}}r)r = 0$$

Integrate both sides with respect to r from 0 to a.

$$\int_0^a c\pi \sum_{m=1}^{\infty} \sqrt{\lambda_{nm}} C_{nm} J_n(\sqrt{\lambda_{nm}}r) J_n(\sqrt{\lambda_{nq}}r) r\, dr = 0$$

Bring the constants in front.

$$c\pi \sum_{m=1}^{\infty} \sqrt{\lambda_{nm}} C_{nm} \int_0^a J_n(\sqrt{\lambda_{nm}}r) J_n(\sqrt{\lambda_{nq}}r) r\, dr = 0$$

Because the Bessel functions are orthogonal with respect to the weight r, the integral is zero if $m \neq q$. As a result, every term in the infinite series vanishes except for the $m = q$ one.

$$c\pi \sqrt{\lambda_{nm}} C_{nm} \int_0^a J_n^2(\sqrt{\lambda_{nm}}r) r\, dr = 0$$

This integral is known.

$$c\pi \sqrt{\lambda_{nm}} C_{nm} \left[\frac{a^2}{2} J_{n+1}^2(\sqrt{\lambda_{nm}}a)\right] = 0$$

So then

$$\boxed{C_{nm} = 0.}$$

To find D_{nm}, start by multiplying both sides of equation (2) by $\sin p\theta$, where p is a positive integer.

$$\sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r) \sqrt{\lambda_{0m}} c C_{0m} \sin p\theta \\ + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r) \sqrt{\lambda_{nm}} c (C_{nm} \cos n\theta \sin p\theta + D_{nm} \sin n\theta \sin p\theta) = \phi(r) \sin p\theta$$

Integrate both sides with respect to θ from 0 to 2π.

$$\int_0^{2\pi} \left[\sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r) \sqrt{\lambda_{0m}} c C_{0m} \sin p\theta \right. \\ \left. + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r) \sqrt{\lambda_{nm}} c (C_{nm} \cos n\theta \sin p\theta + D_{nm} \sin n\theta \sin p\theta)\right] d\theta = \int_0^{2\pi} \phi(r) \sin p\theta\, d\theta$$

Split up the integrals and bring the constants in front.

$$\sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r) \sqrt{\lambda_{0m}} c C_{0m} \underbrace{\int_0^{2\pi} \sin p\theta\, d\theta}_{=0} \\ + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r) \sqrt{\lambda_{nm}} c \left(C_{nm} \underbrace{\int_0^{2\pi} \cos n\theta \sin p\theta\, d\theta}_{=0} + D_{nm} \int_0^{2\pi} \sin n\theta \sin p\theta\, d\theta \right) \\ = \phi(r) \underbrace{\int_0^{2\pi} \sin p\theta\, d\theta}_{=0}$$

Evaluate the integrals.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}cD_{nm}\int_0^{2\pi}\sin n\theta\sin p\theta\,d\theta = 0$$

Because the sine functions are orthogonal, the remaining integral is zero if $n \neq p$. As a result, every term in the infinite series over n vanishes except for the $n = p$ one.

$$\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}cD_{nm}\int_0^{2\pi}\sin^2 n\theta\,d\theta = 0$$

Evaluate the integral.

$$\sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\sqrt{\lambda_{nm}}cD_{nm}(\pi) = 0$$

Multiply both sides by $J_n(\sqrt{\lambda_{nq}}r)r$, where q is a positive integer.

$$c\pi\sum_{m=1}^{\infty}\sqrt{\lambda_{nm}}D_{nm}J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nq}}r)r = 0$$

Integrate both sides with respect to r from 0 to a.

$$\int_0^a c\pi\sum_{m=1}^{\infty}\sqrt{\lambda_{nm}}D_{nm}J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nq}}r)r\,dr = 0$$

Bring the constants in front.

$$c\pi\sum_{m=1}^{\infty}\sqrt{\lambda_{nm}}D_{nm}\int_0^a J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{nq}}r)r\,dr = 0$$

Because the Bessel functions are orthogonal with respect to the weight r, the integral is zero if $m \neq q$. As a result, every term in the infinite series vanishes except for the $m = q$ one.

$$c\pi\sqrt{\lambda_{nm}}D_{nm}\int_0^a J_n^2(\sqrt{\lambda_{nm}}r)r\,dr = 0$$

This integral is known.

$$c\pi\sqrt{\lambda_{nm}}D_{nm}\left[\frac{a^2}{2}J_{n+1}^2(\sqrt{\lambda_{nm}}a)\right] = 0$$

So then

$$\boxed{D_{nm} = 0.}$$