Exercise 4

Find all the solutions of the wave equation of the form $u = e^{-i\omega t} f(r)$ that are finite at the origin, where $r = \sqrt{x^2 + y^2}$.

Solution

The two-dimensional wave equation is

$$u_{tt} = c^2 \nabla^2 u.$$

Expand the Laplacian operator in polar coordinates.

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right)$$

Substitute $u = e^{-i\omega t} f(r)$ into the PDE.

$$\frac{\partial^2}{\partial t^2}[e^{-i\omega t} f(r)] = c^2 \left[\frac{\partial^2}{\partial r^2}[e^{-i\omega t} f(r)] + \frac{1}{r}\frac{\partial}{\partial r}[e^{-i\omega t} f(r)] + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[e^{-i\omega t} f(r)]}_{=0} \right]$$

$$(-i\omega)^2 e^{-i\omega t} f(r) = c^2 e^{-i\omega t} \left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} \right)$$

Divide both sides by $e^{-i\omega t}$.

$$-\omega^2 f(r) = c^2 \left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} \right)$$

Bring $\omega^2 f(r)$ to the right side and then multiply both sides by r^2/c^2.

$$r^2 \frac{d^2 f}{dr^2} + r\frac{df}{dr} + \frac{\omega^2}{c^2} r^2 f = 0$$

This is the parametric form of Bessel's equation of order zero. The general solution is written in terms of zero-order Bessel functions of the first and second kind, J_0 and Y_0, respectively.

$$f(r) = C_1 J_0 \left(\frac{\omega}{c} r \right) + C_2 Y_0 \left(\frac{\omega}{c} r \right)$$

Y_0 diverges if its argument is zero, so we require that $C_2 = 0$.

$$f(r) = C_1 J_0 \left(\frac{\omega}{c} r \right)$$

Therefore,

$$u(r,t) = A e^{-i\omega t} J_0 \left(\frac{\omega}{c} r \right),$$

where A is an arbitrary constant.