Exercise 5

Solve the diffusion equation in the disk of radius a, with $u = B$ on the boundary and $u = 0$ when $t = 0$, where B is a constant. (*Hint:* The answer is radial.)

Solution

The initial boundary value problem to solve is

$$\begin{aligned} u_t = k\nabla^2 u = k\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad 0 \le r \le a,\ 0 \le \theta \le 2\pi \\ u(a, \theta, t) = B \\ u(r, \theta, 0) = 0. \end{aligned}$$

Since the boundary condition is independent of θ, we assume that the solution is as well.

$$u = u(r, t)$$

The angular derivative in the PDE vanishes as a result.

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r\right)$$

To make the boundary condition homogeneous, make the following change of variables.

$$v(r, t) = u(r, t) - B$$

Find the derivatives of u in terms of this new variable.

$$\begin{aligned} u_t &= v_t \\ u_r &= v_r \\ u_{rr} &= v_{rr} \end{aligned}$$

Consequently, v satisfies the diffusion equation as well.

$$v_t = k\left(v_{rr} + \frac{1}{r}v_r\right)$$

The conditions associated with this equation are

$$\begin{aligned} v(a, t) &= u(a, t) - B = B - B = 0 \\ v(r, 0) &= u(r, 0) - B = 0 - B = -B. \end{aligned}$$

Because the PDE for v and its boundary condition are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $v = R(r)T(t)$ and substitute it into the PDE

$$v_t = k\left(v_{rr} + \frac{1}{r}v_r\right) \quad \to \quad \frac{\partial}{\partial t}[R(r)T(t)] = k\left[\frac{\partial^2}{\partial r^2}[R(r)T(t)] + \frac{1}{r}\frac{\partial}{\partial r}[R(r)T(t)]\right]$$

and the boundary condition.

$$v(a, t) = 0 \quad \to \quad R(a)T(t) = 0 \quad \to \quad R(a) = 0$$

Proceed to separate variables in the PDE.

$$R\frac{dT}{dt} = kT\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right)$$

Divide both sides by $kR(r)T(t)$.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right)}_{\text{function of } r}$$

The only way a function of t can be equal to a function of r is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in t and one in r.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} &= \lambda \\ \frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) &= \lambda \end{aligned}\right\}$$

Start by solving the one for R. Multiply both sides of it by r^2R and bring all terms to the left side.

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda r^2 R = 0$$

Suppose first that λ is positive: $\lambda = \beta^2$.

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \beta^2 r^2 R = 0$$

The general solution is written in terms of I_0 and K_0, zero-order modified Bessel functions of the first and second kind, respectively.

$$R(r) = C_1 I_0(\beta r) + C_2 K_0(\beta r)$$

K_0 diverges as $r \to 0$, so we require $C_2 = 0$.

$$R(r) = C_1 I_0(\beta r)$$

Apply the boundary condition now.

$$R(a) = C_1 I_0(\beta a) = 0$$

No nonzero value of β satisfies this equation, so C_1 must be zero. The trivial solution $R(r) = 0$ results, so there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for R becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = 0.$$

Multiply both sides by r.

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} = 0$$

Write the left side as a derivative using the chain rule.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = C_3$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{C_3}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = C_3 \ln r + C_4$$

The logarithm diverges as $r \to 0$, so we require that $C_3 = 0$.

$$R(r) = C_4$$

Apply the boundary condition to determine C_4.

$$R(a) = C_4 = 0$$

The trivial solution $R(r) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\gamma^2$. The ODE for R becomes

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \gamma^2 r^2 R = 0$$

The general solution is written in terms of J_0 and Y_0, zero-order Bessel functions of the first and second kind, respectively.

$$R(r) = C_5 J_0(\gamma r) + C_6 Y_0(\gamma r)$$

Y_0 diverges as $r \to 0$, so we require that $C_6 = 0$.

$$R(r) = C_5 J_0(\gamma r)$$

Apply the boundary condition now.

$$R(a) = C_5 J_0(\gamma a) = 0$$

To avoid the trivial solution, we insist that $C_5 \neq 0$.

$$\begin{aligned} J_0(\gamma a) &= 0 \\ \gamma a &= \alpha_{0n}, \quad n = 1, 2, \ldots \\ \gamma_n &= \frac{\alpha_{0n}}{a} \end{aligned}$$

Here α_{0n} is the nth positive zero of J_0. The negative eigenvalues are $\lambda = -\alpha_{0n}^2/a^2$, and the eigenfunctions associated with them are

$$R(r) = C_5 J_0(\gamma r) \quad \rightarrow \quad R_n(r) = J_0\left(\frac{\alpha_{0n}}{a}r\right).$$

With this formula for λ, solve the ODE for T now.

$$\frac{dT}{dt} = -\frac{k\alpha_{0n}^2}{a^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-\frac{k\alpha_{0n}^2}{a^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for v is a linear combination of the eigenfunctions $R(r)T(t)$ over all the eigenvalues.

$$v(r,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{k\alpha_{0n}^2}{a^2}t\right) J_0\left(\frac{\alpha_{0n}}{a}r\right)$$

Use the initial condition $v(r,0) = -B$ now to determine the coefficients A_n.

$$v(r,0) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_{0n}}{a}r\right) = -B$$

Multiply both sides by $J_0(\alpha_{0m}r/a)r$, where m is a positive integer,

$$\sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_{0n}}{a}r\right) J_0\left(\frac{\alpha_{0m}}{a}r\right) r = -BJ_0\left(\frac{\alpha_{0m}}{a}r\right) r$$

and then integrate both sides with respect to r from 0 to a.

$$\int_0^a \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_{0n}}{a}r\right) J_0\left(\frac{\alpha_{0m}}{a}r\right) r\,dr = \int_0^a (-B)J_0\left(\frac{\alpha_{0m}}{a}r\right) r\,dr$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} A_n \int_0^a J_0\left(\frac{\alpha_{0n}}{a}r\right) J_0\left(\frac{\alpha_{0m}}{a}r\right) r\,dr = -B\int_0^a J_0\left(\frac{\alpha_{0m}}{a}r\right) r\,dr$$

Because the Bessel functions are orthogonal with respect to the weight r, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$A_n \int_0^a J_0^2\left(\frac{\alpha_{0n}}{a}r\right) r\,dr = -B\int_0^a J_0\left(\frac{\alpha_{0n}}{a}r\right) r\,dr$$

Evaluate the integrals.

$$A_n\left[\frac{a^2}{2}J_1^2(\alpha_{0n})\right] = -B\left[\frac{a^2}{\alpha_{0n}}J_1(\alpha_{0n})\right]$$

Solve for A_n.

$$A_n = -\frac{2B}{\alpha_{0n}J_1(\alpha_{0n})}$$

So then

$$v(r,t) = \sum_{n=1}^{\infty} \left[-\frac{2B}{\alpha_{0n} J_1(\alpha_{0n})}\right] \exp\left(-\frac{k\alpha_{0n}^2}{a^2} t\right) J_0\left(\frac{\alpha_{0n}}{a} r\right).$$

Therefore, since $u(r,t) = B + v(r,t)$,

$$u(r,t) = B - 2B \sum_{n=1}^{\infty} \frac{1}{\alpha_{0n} J_1(\alpha_{0n})} \exp\left(-\frac{k\alpha_{0n}^2}{a^2} t\right) J_0\left(\frac{\alpha_{0n}}{a} r\right).$$

This answer is in disagreement with the one at the back of the book,

$$u = B + 2B \textstyle\sum_{n=1}^{\infty} [\beta_n J_1(\beta_n)]^{-1} e^{-\beta_n^2 kt/a} J_0(\beta_n r/a), \text{ where } J_0(\beta_n) = 0,$$

which neither satisfies the PDE nor the initial condition. Note that $\beta_n = \alpha_{0n}$.