Exercise 7

Let D be the semidisk $\{x^2 + y^2 < b^2,\ y > 0\}$. Consider the diffusion equation in D with the conditions: $u = 0$ on bdy D and $u = \phi(r, \theta)$ when $t = 0$. Write the complete expansion for the solution $u(r, \theta, t)$, including the formulas for the coefficients.

Solution

The initial boundary value problem to solve is as follows.

$$\begin{aligned} u_t = k\nabla^2 u = k\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad 0 \le r \le b,\ 0 \le \theta \le \pi \\ u(r, 0, t) = 0 \\ u(r, \pi, t) = 0 \\ u(b, \theta, t) = 0 \\ u(r, \theta, 0) = \phi(r, \theta) \end{aligned}$$

Since the PDE and the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$ and substitute it into the PDE

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \quad \rightarrow \quad R\Theta T' = k\left(R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T\right)$$

and the boundary conditions.

$$\begin{aligned} u(r, 0, t) = 0 \quad &\rightarrow \quad R(r)\Theta(0)T(t) = 0 \quad \rightarrow \quad \Theta(0) = 0 \\ u(r, \pi, t) = 0 \quad &\rightarrow \quad R(r)\Theta(\pi)T(t) = 0 \quad \rightarrow \quad \Theta(\pi) = 0 \\ u(b, \theta, t) = 0 \quad &\rightarrow \quad R(b)\Theta(\theta)T(t) = 0 \quad \rightarrow \quad R(b) = 0 \end{aligned}$$

Proceed to separate variables in the PDE by dividing both sides by $kR\Theta T$.

$$\underbrace{\frac{T'}{kT}}_{\text{function of } t} = \underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{function of } r \text{ and } \theta}$$

The only way a function of t can be equal to a function of r and θ is if both are equal to a constant λ.

$$\frac{T'}{kT} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda$$

Separate variables in the second equation.

$$\underbrace{r^2\frac{R''}{R} + r\frac{R'}{R} - \lambda r^2}_{\text{function of } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{function of } \theta}$$

The only way a function of r can be equal to a function of θ is if both are equal to another constant μ.

$$r^2\frac{R''}{R} + r\frac{R'}{R} - \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu$$

As a result of applying the method of separation of variables, the PDE has been reduced to three ODEs—one in t, one in r, and one in θ.

$$\left.\begin{aligned} \frac{T'}{kT} &= \lambda \\ -\frac{\Theta''}{\Theta} &= \mu \\ r^2\frac{R''}{R} + r\frac{R'}{R} - \lambda r^2 &= \mu \end{aligned}\right\}$$

Start by solving the one for Θ. Suppose first that μ is positive: $\mu = \beta^2$.

$$\Theta'' = -\beta^2\Theta$$

The general solution is written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\beta\theta + C_2 \sin\beta\theta$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} \Theta(0) &= C_1 = 0 \\ \Theta(\pi) &= C_1\cos\beta\pi + C_2\sin\beta\pi = 0 \end{aligned}$$

The second equation reduces to $C_2 \sin\beta\pi = 0$. To avoid the trivial solution, we insist that $C_2 \neq 0$. Then

$$\begin{aligned} \sin\beta\pi &= 0 \\ \beta\pi &= m\pi, \quad m = 1, 2, \ldots \\ \beta_m &= m. \end{aligned}$$

The positive eigenvalues are $\mu = m^2$, and the eigenfunctions associated with them are

$$\begin{aligned} \Theta(\theta) &= C_1\cos\beta\theta + C_2\sin\beta\theta \\ &= C_2\sin\beta\theta \quad \rightarrow \quad \Theta_m(\theta) = \sin m\theta. \end{aligned}$$

With this formula for μ, solve the ODE for R now.

$$r^2R'' + rR' + (-\lambda r^2 - m^2)R = 0$$

If λ is not negative, then the solution will be in terms of functions that cannot satisfy the boundary conditions, $R(b) = 0$ and $R(0) =$ finite. Assuming that λ is negative then, $\lambda = -\gamma^2$, we get

$$r^2R'' + rR' + (\gamma^2r^2 - m^2)R = 0.$$

This is the parametric form of Bessel's equation of order m. The general solution is in terms of mth-order Bessel functions of the first and second kind, J_m and Y_m, respectively.

$$R(r) = C_3J_m(\gamma r) + C_4Y_m(\gamma r)$$

Y_m diverges as $r \to 0$, so we require that $C_4 = 0$ to satisfy $R(0) =$ finite.

$$R(r) = C_3J_m(\gamma r)$$

Apply the other boundary condition now.

$$R(b) = C_3 J_m(\gamma b) = 0$$

To avoid the trivial solution, we insist that $C_3 \neq 0$. Then

$$\begin{aligned} J_m(\gamma b) &= 0 \\ \gamma b &= \alpha_{mn}, \quad n = 1, 2, \ldots \\ \gamma_{mn} &= \frac{\alpha_{mn}}{b}. \end{aligned}$$

Here α_{mn} represents the nth positive zero of J_m. The eigenfunctions associated with $\lambda = -\alpha_{mn}^2/b^2$ are

$$\begin{aligned} R(r) &= C_3 J_m(\gamma r) + C_4 Y_m(\gamma r) \\ &= C_3 J_m(\gamma r) \quad \rightarrow \quad R_{mn}(r) = J_m\left(\frac{\alpha_{mn}}{b} r\right). \end{aligned}$$

With this formula for λ, solve the ODE for T now.

$$T' = -\frac{k\alpha_{mn}^2}{b^2} T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_5 \exp\left(-\frac{k\alpha_{mn}^2}{b^2} t\right)$$

Suppose secondly that μ is zero: $\mu = 0$. The ODE for Θ becomes

$$\Theta'' = 0.$$

Integrate both sides with respect to θ twice.

$$\Theta(\theta) = C_6 \theta + C_7$$

Apply the boundary conditions to determine C_6 and C_7.

$$\begin{aligned} \Theta(0) &= C_7 = 0 \\ \Theta(\pi) &= C_6 \pi + C_7 = 0 \end{aligned}$$

The second equation implies that $C_6 = 0$, which results in the trivial solution $\Theta(\theta) = 0$. Thus, zero is not an eigenvalue for μ. Suppose thirdly that μ is negative: $\mu = -\eta^2$. The ODE for Θ becomes

$$\Theta'' = \eta^2 \Theta.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_8 \cosh \eta\theta + C_9 \sinh \eta\theta$$

Apply the boundary conditions to determine C_8 and C_9.

$$\begin{aligned} \Theta(0) &= C_8 = 0 \\ \Theta(\pi) &= C_8 \cosh \eta\pi + C_9 \sinh \eta\pi = 0 \end{aligned}$$

The second equation reduces to $C_9 \sinh \eta\pi = 0$. No nonzero value of η satisfies it, so C_9 must be zero. The trivial solution $\Theta(\theta) = 0$ results, which means there are no negative eigenvalues for μ. According to the principle of superposition, the general solution for u is a linear combination of the eigenfunctions $R(r)\Theta(\theta)T(t)$ over all the eigenvalues.

$$\boxed{u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \exp\left(-\frac{k\alpha_{mn}^2}{b^2}t\right) \sin m\theta J_m\left(\frac{\alpha_{mn}}{b}r\right)}$$

Use the initial condition now to determine A_{mn}.

$$u(r,\theta,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin m\theta J_m\left(\frac{\alpha_{mn}}{b}r\right) = \phi(r,\theta)$$

Multiply both sides by $\sin p\theta$, where p is a positive integer.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin m\theta \sin p\theta J_m\left(\frac{\alpha_{mn}}{b}r\right) = \phi(r,\theta)\sin p\theta$$

Integrate both sides with respect to θ from 0 to π.

$$\int_0^\pi \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin m\theta \sin p\theta J_m\left(\frac{\alpha_{mn}}{b}r\right) d\theta = \int_0^\pi \phi(r,\theta)\sin p\theta\, d\theta$$

Bring the constants in front.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m\left(\frac{\alpha_{mn}}{b}r\right) \int_0^\pi \sin m\theta \sin p\theta\, d\theta = \int_0^\pi \phi(r,\theta)\sin p\theta\, d\theta$$

Because the sine functions are orthogonal, the integral on the left is zero if $m \neq p$. As a result, every term in the infinite series over m vanishes except for the $m = p$ one.

$$\sum_{n=1}^{\infty} A_{mn} J_m\left(\frac{\alpha_{mn}}{b}r\right) \int_0^\pi \sin^2 m\theta\, d\theta = \int_0^\pi \phi(r,\theta)\sin m\theta\, d\theta$$

Evaluate the integral on the left.

$$\sum_{n=1}^{\infty} A_{mn} J_m\left(\frac{\alpha_{mn}}{b}r\right)\left(\frac{\pi}{2}\right) = \int_0^\pi \phi(r,\theta)\sin m\theta\, d\theta$$

Multiply both sides by $J_m(\alpha_{mq}r/b)r$, where q is a positive integer.

$$\frac{\pi}{2}\sum_{n=1}^{\infty} A_{mn} J_m\left(\frac{\alpha_{mn}}{b}r\right) J_m\left(\frac{\alpha_{mq}}{b}r\right) r = \int_0^\pi \phi(r,\theta)\sin m\theta J_m\left(\frac{\alpha_{mq}}{b}r\right) r\, d\theta$$

Integrate both sides with respect to r from 0 to b.

$$\int_0^b \frac{\pi}{2}\sum_{n=1}^{\infty} A_{mn} J_m\left(\frac{\alpha_{mn}}{b}r\right) J_m\left(\frac{\alpha_{mq}}{b}r\right) r\, dr = \int_0^\pi \int_0^b \phi(r,\theta)\sin m\theta J_m\left(\frac{\alpha_{mq}}{b}r\right) r\, dr\, d\theta$$

Bring the constants in front of the integral on the left.

$$\frac{\pi}{2}\sum_{n=1}^{\infty} A_{mn}\int_0^b J_m\left(\frac{\alpha_{mn}}{b}r\right)J_m\left(\frac{\alpha_{mq}}{b}r\right)r\,dr = \int_0^{\pi}\int_0^b \phi(r,\theta)\sin m\theta J_m\left(\frac{\alpha_{mq}}{b}r\right)r\,dr\,d\theta$$

Because the Bessel functions are orthogonal with respect to the weight r, the integral on the left is zero if $n \neq q$. As a result, every term in the infinite series vanishes except for the $n = q$ one.

$$\frac{\pi}{2}A_{mn}\int_0^b J_m^2\left(\frac{\alpha_{mn}}{b}r\right)r\,dr = \int_0^{\pi}\int_0^b \phi(r,\theta)\sin m\theta J_m\left(\frac{\alpha_{mn}}{b}r\right)r\,dr\,d\theta$$

Evaluate the integral on the left.

$$\frac{\pi}{2}A_{mn}\left[\frac{b^2}{2}J_{m+1}^2(\alpha_{mn})\right] = \int_0^{\pi}\int_0^b \phi(r,\theta)\sin m\theta J_m\left(\frac{\alpha_{mn}}{b}r\right)r\,dr\,d\theta$$

Therefore,

$$\boxed{A_{mn} = \frac{4}{\pi b^2 J_{m+1}^2(\alpha_{mn})}\int_0^{\pi}\int_0^b \phi(r,\theta)\sin m\theta J_m\left(\frac{\alpha_{mn}}{b}r\right)r\,dr\,d\theta.}$$