Exercise 2

Verify each entry in the table of properties of Fourier transforms.

Solution

This is the table of properties of Fourier transforms the exercise is referring to (on page 346).

	Function	Transform
(i)	$\frac{df}{dx}$	$ikF(k)$
(ii)	$xf(x)$	$i\frac{dF}{dk}$
(iii)	$f(x-a)$	$e^{-iak}F(k)$
(iv)	$e^{iax}f(x)$	$F(k-a)$
(v)	$af(x)+bg(x)$	$aF(k)+bG(k)$
(vi)	$f(ax)$	$\frac{1}{\|a\|}F\left(\frac{k}{a}\right) \quad (a \neq 0)$

We will verify these transforms using the definitions of the Fourier transform,

$$\mathcal{F}\{f(x)\} = F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx,$$

and inverse Fourier transform,

$$\mathcal{F}^{-1}\{F(k)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk.$$

Verification of Property i

$$\mathcal{F}\left\{\frac{df}{dx}\right\} = \int_{-\infty}^{\infty} \frac{df}{dx}e^{-ikx}\,dx$$

Use integration by parts.

$$= f(x)e^{-ikx}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\frac{d}{dx}(e^{-ikx})\,dx$$

$f(x)$ is assumed to tend to zero as $x \to \pm\infty$, so the first term is zero.

$$\begin{aligned} &= -\int_{-\infty}^{\infty} f(x)(-ik)e^{-ikx}\,dx \\ &= ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \\ &= ikF(k) \end{aligned}$$

This verifies the first property.

Verification of Property ii

$$\begin{aligned}\mathcal{F}\{xf(x)\} &= \int_{-\infty}^{\infty} xf(x)e^{-ikx}\,dx \\ &= \int_{-\infty}^{\infty} xf(x)\frac{d}{dk}\left(\frac{e^{-ikx}}{-ix}\right)dx \\ &= \frac{1}{-i}\int_{-\infty}^{\infty} f(x)\frac{d}{dk}(e^{-ikx})\,dx \\ &= i\frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \\ &= i\frac{dF}{dk}\end{aligned}$$

This verifies the second property.

Verification of Property iii

$$\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)e^{-ikx}\,dx$$

Make the following substitution.

$$\begin{aligned} y &= x - a \quad \rightarrow \quad y + a = x \\ dy &= dx \end{aligned}$$

As a result,

$$\begin{aligned}\mathcal{F}\{f(x-a)\} &= \int_{-\infty}^{\infty} f(y)e^{-ik(y+a)}\,dx \\ &= \int_{-\infty}^{\infty} f(y)e^{-iky-ika}\,dx \\ &= e^{-ika}\int_{-\infty}^{\infty} f(y)e^{-iky}\,dx \\ &= e^{-iak}F(k).\end{aligned}$$

This verifies the third property.

Verification of Property iv

$$\begin{aligned}\mathcal{F}\left\{e^{iax}f(x)\right\} &= \int_{-\infty}^{\infty} e^{iax}f(x)e^{-ikx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-ikx+iax}\,dx = \int_{-\infty}^{\infty} f(x)e^{-i(k-a)x}\,dx \\ &= F(k-a)\end{aligned}$$

This verifies the fourth property.

Verification of Property v

$$\begin{aligned}
\mathcal{F}\{af(x)+bg(x)\} &= \int_{-\infty}^{\infty} [af(x)+bg(x)]e^{-ikx}\,dx \\
&= \int_{-\infty}^{\infty} [af(x)e^{-ikx}+bg(x)e^{-ikx}]\,dx \\
&= \int_{-\infty}^{\infty} af(x)e^{-ikx}\,dx + \int_{-\infty}^{\infty} bg(x)e^{-ikx}\,dx \\
&= a\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx + b\int_{-\infty}^{\infty} g(x)e^{-ikx}\,dx \\
&= aF(k)+bG(k)
\end{aligned}$$

This verifies the fifth property.

Verification of Property vi

$$\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)e^{-ikx}\,dx$$

Make the following substitution, assuming a to be positive.

$$\begin{aligned}
y &= ax & \quad\to\quad & \frac{y}{a} = x \\
dy &= a\,dx & \quad\to\quad & \frac{dy}{a} = dx
\end{aligned}$$

Consequently,

$$\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(y)e^{-iky/a}\left(\frac{dy}{a}\right) = \frac{1}{a}\int_{-\infty}^{\infty} f(y)\exp\left(-i\frac{k}{a}y\right)dy = \frac{1}{a}F\left(\frac{k}{a}\right).$$

Make the aforementioned substitution again, assuming a to be negative now.

$$\mathcal{F}\{f(ax)\} = \int_{\infty}^{-\infty} f(y)e^{-iky/a}\left(\frac{dy}{a}\right) = -\frac{1}{a}\int_{-\infty}^{\infty} f(y)\exp\left(-i\frac{k}{a}y\right)dy = -\frac{1}{a}F\left(\frac{k}{a}\right)$$

Therefore,

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}F\left(\frac{k}{a}\right).$$