Exercise 3

Show that

$$\frac{1}{2\pi^2 cr}\int_0^\infty \sin kct \sin kr\,dk = \frac{1}{8\pi^2 cr}\int_{-\infty}^\infty [e^{ik(ct-r)} - e^{ik(ct+r)}]\,dk$$
$$= \frac{1}{4\pi cr}[\delta(ct-r) - \delta(ct+r)].$$

Solution

Because the integrand is even in k, the interval of integration can be extended to $(-\infty,\infty)$ by multiplying the integral by a factor of $1/2$.

$$\frac{1}{2\pi^2 cr}\int_0^\infty \sin kct \sin kr\,dk = \frac{1}{4\pi^2 cr}\int_{-\infty}^\infty \sin kct \sin kr\,dk$$

Use Euler's formula to represent the sines in terms of exponential functions.

$$= \frac{1}{4\pi^2 cr}\int_{-\infty}^\infty \left(\frac{e^{ikct}-e^{-ikct}}{2i}\right)\left(\frac{e^{ikr}-e^{-ikr}}{2i}\right)dk$$
$$= -\frac{1}{16\pi^2 cr}\int_{-\infty}^\infty (e^{ikct}-e^{-ikct})(e^{ikr}-e^{-ikr})\,dk$$
$$= -\frac{1}{16\pi^2 cr}\int_{-\infty}^\infty (e^{ikct+ikr}-e^{ikct-ikr}-e^{-ikct+ikr}+e^{-ikct-ikr})\,dk$$
$$= \frac{1}{16\pi^2 cr}\int_{-\infty}^\infty [-e^{ik(ct+r)}+e^{ik(ct-r)}+e^{ik(-ct+r)}-e^{ik(-ct-r)}]\,dk$$
$$= \frac{1}{16\pi^2 cr}\left[-\int_{-\infty}^\infty e^{ik(ct+r)}\,dk + \int_{-\infty}^\infty e^{ik(ct-r)}\,dk + \int_{-\infty}^\infty e^{ik(-ct+r)}\,dk - \int_{-\infty}^\infty e^{ik(-ct-r)}\,dk\right]$$
$$= \frac{1}{16\pi^2 cr}\left[-\frac{1}{2\pi}\int_{-\infty}^\infty 2\pi e^{ik(ct+r)}\,dk + \frac{1}{2\pi}\int_{-\infty}^\infty 2\pi e^{ik(ct-r)}\,dk + \frac{1}{2\pi}\int_{-\infty}^\infty 2\pi e^{ik(-ct+r)}\,dk - \frac{1}{2\pi}\int_{-\infty}^\infty 2\pi e^{ik(-ct-r)}\,dk\right]$$
$$= \frac{1}{16\pi^2 cr}[-2\pi\delta(ct+r)+2\pi\delta(ct-r)+2\pi\delta(-ct+r)-2\pi\delta(-ct-r)]$$
$$= \frac{1}{8\pi cr}[-\delta(ct+r)+\delta(ct-r)+\delta(-ct+r)-\delta(-ct-r)]$$

The argument of $\delta(ct+r)$ is equal to 0 when $ct=-r$, which is the same as for $\delta(-ct-r)$. Also, the argument of $\delta(ct-r)$ is equal to 0 when $ct=r$, which is the same as for $\delta(-ct+r)$.

$$= \frac{1}{8\pi cr}[-2\delta(ct+r)+2\delta(ct-r)]$$

Therefore,

$$\frac{1}{2\pi^2 cr}\int_0^\infty \sin kct \sin kr\,dk = \frac{1}{4\pi cr}[\delta(ct-r)-\delta(ct+r)] = \frac{1}{8\pi^2 cr}[2\pi\delta(ct-r)-2\pi\delta(ct+r)]$$
$$= \frac{1}{8\pi^2 cr}\int_{-\infty}^\infty [e^{ik(ct-r)}-e^{ik(ct+r)}]\,dk.$$