Exercise 4

Prove the following properties of the convolution.

(a) $f * g = g * f$.

(b) $(f * g)' = f' * g = f * g'$, where $'$ denotes the derivative in one variable.

(c) $f * (g * h) = (f * g) * h$.

Solution

The convolution of two functions, $f(x)$ and $g(x)$, is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

Part (a)

Make the following substitution in the integral.

$$\begin{aligned} s &= x - y & &\rightarrow & y &= x - s \\ ds &= -dy & &\rightarrow & -\,ds &= dy \end{aligned}$$

Consequently,

$$\begin{aligned} f * g &= \int_{\infty}^{-\infty} f(s)g(x-s)(-ds) \\ &= \int_{-\infty}^{\infty} g(x-s)f(s)\,ds \\ &= g * f \end{aligned}$$

Part (b)

$$(f * g)' = \frac{d}{dx}\int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} \frac{d}{dx}[f(x-y)g(y)]\,dy = \int_{-\infty}^{\infty} f'(x-y)g(y)\,dy = f' * g$$

Also,

$$(f * g)' = (g * f)' = \frac{d}{dx}\int_{-\infty}^{\infty} g(x-y)f(y)\,dy = \int_{-\infty}^{\infty} \frac{d}{dx}[g(x-y)f(y)]\,dy = \int_{-\infty}^{\infty} g'(x-y)f(y)\,dy.$$

Make the following substitution in the integral.

$$\begin{aligned} s &= x - y & &\rightarrow & y &= x - s \\ ds &= -dy & &\rightarrow & -\,ds &= dy \end{aligned}$$

Consequently,

$$(f * g)' = \int_{\infty}^{-\infty} g'(s)f(x-s)(-ds) = \int_{-\infty}^{\infty} f(x-s)g'(s)\,ds = f * g'.$$

Part (c)

$$\begin{aligned} f*(g*h) &= \int_{-\infty}^{\infty} f(x-s)\left[\int_{-\infty}^{\infty} g(s-y)h(y)\,dy\right] ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-s)g(s-y)h(y)\,dy\,ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-s)g(s-y)h(y)\,ds\,dy \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-s)g(s-y)\,ds\right] h(y)\,dy \end{aligned}$$

Make the following substitution in the integral.

$$\begin{aligned} p &= s-y \quad \rightarrow \quad s = y+p \\ dp &= ds \end{aligned}$$

Consequently,

$$\begin{aligned} f*(g*h) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-y-p)g(p)\,dp\right] h(y)\,dy \\ &= (f*g)*h. \end{aligned}$$