Exercise 9

Use Fourier transforms to solve the ODE $-u_{xx} + a^2u = \delta$, where $\delta = \delta(x)$ is the delta function.

Solution

Since the ODE is linear and the domain is assumed to be $-\infty < x < \infty$, a Fourier transform can be applied to solve it. Here we define the Fourier transform of a function $u(x)$ as

$$\mathcal{F}\{u(x)\} = U(k) = \int_{-\infty}^{\infty} u(x)e^{-ikx}\,dx.$$

As a result, the derivative of u with respect to x transforms as follows.

$$\mathcal{F}\left\{\frac{d^n u}{dx^n}\right\} = (ik)^n U(k)$$

Take the Fourier transform of both sides of the ODE.

$$\mathcal{F}\{-u_{xx} + a^2u\} = \mathcal{F}\{\delta(x)\}$$

Use the fact that the operator is linear on the left side, and use the definition on the right side.

$$-\mathcal{F}\{u_{xx}\} + a^2\mathcal{F}\{u\} = \int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx$$

Transform the second derivative with the expression above.

$$-(ik)^2U(k) + a^2U(k) = e^{-ik\cdot 0}$$

Factor $U(k)$ on the left side.

$$(k^2 + a^2)U(k) = 1$$

Thus,

$$U(k) = \frac{1}{k^2 + a^2}.$$

Now that we have $U(k)$, we can obtain $u(x)$ by taking the inverse Fourier transform of it.

$$\begin{aligned} u(x) &= \mathcal{F}^{-1}\{U(k)\} \\ &= \mathcal{F}^{-1}\left\{\frac{1}{k^2 + a^2}\right\} \end{aligned}$$

Equation (7) on page 345 in the text gives a Fourier transform pair that involves this function.

$$\mathcal{F}\{e^{-a|x|}\} = \frac{2a}{a^2 + k^2} \quad (a > 0) \tag{7}$$

Divide both sides by $2a$ and then take the inverse Fourier transform of both sides.

$$\frac{1}{2a}\mathcal{F}\{e^{-a|x|}\} = \frac{1}{a^2 + k^2} \quad \rightarrow \quad \frac{1}{2a}e^{-a|x|} = \mathcal{F}^{-1}\left\{\frac{1}{a^2 + k^2}\right\}$$

Therefore,

$$u(x) = \frac{1}{2a}e^{-a|x|}.$$