Exercise 1

Use the Fourier transform directly to solve the heat equation with a convection term, namely, $u_t = \kappa u_{xx} + \mu u_x$ for $-\infty < x < \infty$, with an initial condition $u(x,0) = \phi(x)$, assuming that $u(x,t)$ is bounded and $\kappa > 0$.

Solution

Since the PDE is linear and the x variable goes from $-\infty$ to ∞, the Fourier transform can be applied to solve it. Here we define the Fourier transform of a function $u(x,t)$ as

$$\mathcal{F}\{u(x,t)\} = U(\omega,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx.$$

(ω is used rather than k to avoid confusion with κ in the PDE.) As a result, the derivatives of u with respect to x and t transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (i\omega)^n U(\omega,t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n U}{dt^n}$$

Take the Fourier transform of both sides of the PDE

$$\mathcal{F}\{u_t\} = \mathcal{F}\{\kappa u_{xx} + \mu u_x\}$$

and its initial condition.

$$\mathcal{F}\{u(x,0)\} = \mathcal{F}\{\phi(x)\} \quad \rightarrow \quad U(\omega,0) = \Phi(\omega) \tag{1}$$

Use the fact that the Fourier transform is a linear operator.

$$\mathcal{F}\{u_t\} = \kappa\mathcal{F}\{u_{xx}\} + \mu\mathcal{F}\{u_x\}$$

Transform the derivatives with the expressions above.

$$\frac{dU}{dt} = \kappa(i\omega)^2 U(\omega,t) + \mu(i\omega)U(\omega,t)$$

The second-order PDE has thus been reduced to a first-order ODE that can be solved with separation of variables.

$$\frac{dU}{dt} = (-\kappa\omega^2 + i\mu\omega)U$$

Separate variables.

$$\frac{dU}{U} = (-\kappa\omega^2 + i\mu\omega)\,dt$$

Integrate both sides.

$$\ln|U| = (-\kappa\omega^2 + i\mu\omega)t + C$$

Exponentiate both sides.

$$|U| = e^C e^{(-\kappa\omega^2 + i\mu\omega)t}$$

Remove the absolute value sign by introducing $\pm$ on the right side.

$$U(\omega,t) = \pm e^C e^{(-\kappa\omega^2 + i\mu\omega)t}$$

Use a new constant of integration.

$$U(\omega,t) = A(\omega)e^{(-\kappa\omega^2 + i\mu\omega)t}$$

Use the Fourier-transformed initial condition in equation (1) to determine $A(\omega)$.

$$U(\omega,0) = A(\omega) = \Phi(\omega)$$

So we have

$$U(\omega,t) = \Phi(\omega)e^{(-\kappa\omega^2 + i\mu\omega)t}.$$

Now that we have $U(\omega,t)$, we can change back to $u(x,t)$ by taking the inverse Fourier transform of it.

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\{U(\omega,t)\} \\ &= \mathcal{F}^{-1}\{\Phi(\omega)e^{(-\kappa\omega^2 + i\mu\omega)t}\} \end{aligned}$$

Because we are taking the inverse Fourier transform of a product of two functions, the convolution theorem can be applied, which states

$$\mathcal{F}^{-1}\{\Phi(\omega)G(\omega)\} = \int_{-\infty}^{\infty} \phi(x-s)g(s)\,ds = \int_{-\infty}^{\infty} \phi(s)g(x-s)\,ds.$$

All that we have to do then is calculate g, the inverse Fourier transform of the exponential function, and we can write the solution using the convolution theorem.

$$\mathcal{F}^{-1}\{e^{(-\kappa\omega^2 + i\mu\omega)t}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(-\kappa\omega^2 + i\mu\omega)t} e^{i\omega x}\,d\omega$$

Combine the exponential functions and then proceed to complete the square in the exponent.

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[(-\kappa\omega^2 + i\mu\omega)t + i\omega x\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\kappa t\omega^2 + i(\mu t + x)\omega\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\kappa t\left[\omega^2 - \frac{i(\mu t + x)}{\kappa t}\omega\right]\right\} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\kappa t\left[\omega^2 - \frac{i(\mu t + x)}{\kappa t}\omega + \frac{i^2(\mu t + x)^2}{4\kappa^2 t^2}\right] + \kappa t\frac{i^2(\mu t + x)^2}{4\kappa^2 t^2}\right\} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\kappa t\left[\omega - \frac{i(\mu t + x)}{2\kappa t}\right]^2 - \frac{(\mu t + x)^2}{4\kappa t}\right\} d\omega \\ &= \frac{1}{2\pi}\exp\left[-\frac{(\mu t + x)^2}{4\kappa t}\right]\int_{-\infty}^{\infty} \exp\left\{-\kappa t\left[\omega - \frac{i(\mu t + x)}{2\kappa t}\right]^2\right\} d\omega \end{aligned}$$

Make the following substitution in the integral.

$$p = \omega - \frac{i(\mu t + x)}{2\kappa t}$$
$$dp = d\omega$$

We obtain

$$\mathcal{F}^{-1}\{e^{(-\kappa\omega^2 + i\mu\omega)t}\} = \frac{1}{2\pi} \exp\left[-\frac{(\mu t + x)^2}{4\kappa t}\right] \int_{-\infty}^{\infty} e^{-\kappa t p^2}\, dp.$$

Make another substitution.

$$r = \sqrt{\kappa t}\, p \quad \rightarrow \quad r^2 = \kappa t p^2$$
$$dr = \sqrt{\kappa t}\, dp \quad \rightarrow \quad \frac{1}{\sqrt{\kappa t}}\, dr = dp$$

We obtain

$$\begin{aligned}
\mathcal{F}^{-1}\{e^{(-\kappa\omega^2 + i\mu\omega)t}\} &= \frac{1}{2\pi} \exp\left[-\frac{(\mu t + x)^2}{4\kappa t}\right] \int_{-\infty}^{\infty} e^{-r^2} \left(\frac{1}{\sqrt{\kappa t}}\, dr\right) \\
&= \frac{1}{2\pi\sqrt{\kappa t}} \exp\left[-\frac{(\mu t + x)^2}{4\kappa t}\right] \int_{-\infty}^{\infty} e^{-r^2}\, dr \\
&= \frac{1}{2\pi\sqrt{\kappa t}} \exp\left[-\frac{(\mu t + x)^2}{4\kappa t}\right] \cdot \sqrt{\pi} \\
&= \frac{1}{2\sqrt{\pi\kappa t}} \exp\left[-\frac{(\mu t + x)^2}{4\kappa t}\right].
\end{aligned}$$

By the convolution theorem then,

$$u(x,t) = \int_{-\infty}^{\infty} \phi(s) \frac{1}{2\sqrt{\pi\kappa t}} \exp\left[-\frac{(\mu t + x - s)^2}{4\kappa t}\right] ds.$$

Therefore,

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \phi(s) \exp\left[-\frac{(\mu t + x - s)^2}{4\kappa t}\right] ds.$$