Exercise 6

Use the Fourier transform to solve $u_{xx} + u_{yy} = 0$ in the infinite strip $\{0 < y < 1,\ -\infty < x < \infty\}$, together with the conditions $u(x, 0) = 0$ and $u(x, 1) = f(x)$.

Solution

Since the PDE is linear and the x variable goes from $-\infty$ to ∞, the Fourier transform can be applied to solve it. Here we define the Fourier transform of a function $u(x, y)$ as

$$\mathcal{F}\{u(x,y)\} = U(k,y) = \int_{-\infty}^{\infty} u(x,y)e^{-ikx}\,dx.$$

As a result, the derivatives of u with respect to x and y transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,y)$$

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} = \frac{d^n U}{dy^n}$$

Take the Fourier transform of both sides of the PDE

$$\mathcal{F}\{u_{xx} + u_{yy}\} = \mathcal{F}\{0\}$$

and the boundary conditions.

$$\mathcal{F}\{u(x,0)\} = \mathcal{F}\{0\} \quad \rightarrow \quad U(k,0) = 0 \tag{1}$$

$$\mathcal{F}\{u(x,1)\} = \mathcal{F}\{f(x)\} \quad \rightarrow \quad U(k,1) = F(k) \tag{2}$$

Use the fact that the Fourier transform is a linear operator.

$$\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_{yy}\} = 0$$

Transform the derivatives with the expressions above.

$$(ik)^2 U(k,y) + \frac{d^2U}{dy^2} = 0$$

The second-order PDE has thus been reduced to a second-order ODE whose solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\frac{d^2U}{dy^2} = k^2 U$$

$$U(k,y) = A(k)\cosh ky + B(k)\sinh ky$$

Apply the Fourier-transformed boundary conditions in equations (1) and (2) to determine $A(k)$ and $B(k)$.

$$U(k,0) = A(k) = 0$$

$$U(k,1) = A(k)\cosh k + B(k)\sinh k = F(k) \quad \rightarrow \quad B(k) = \frac{F(k)}{\sinh k}$$

Thus,

$$U(k,y) = F(k)\frac{\sinh ky}{\sinh k}.$$

Now that we have $U(k,y)$, we can change back to $u(x,y)$ by taking the inverse Fourier transform of it.

$$\begin{aligned} u(x,y) &= \mathcal{F}^{-1}\{U(k,y)\} \\ &= \mathcal{F}^{-1}\left\{F(k)\frac{\sinh ky}{\sinh k}\right\} \end{aligned}$$

Because we are taking the inverse Fourier transform of a product of two functions, the convolution theorem can be applied, which states

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \int_{-\infty}^{\infty} f(x-s)g(s)\,ds = \int_{-\infty}^{\infty} f(s)g(x-s)\,ds.$$

All that we have to do then is calculate g, the inverse Fourier transform of $\sinh ky/\sinh k$.

$$\mathcal{F}^{-1}\left\{\frac{\sinh ky}{\sinh k}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh ky}{\sinh k}e^{ikx}\,dk$$

Use Euler's formula, $e^{ikx} = \cos kx + i\sin kx$, to write the exponential in terms of sine and cosine.

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh ky}{\sinh k}(\cos kx + i\sin kx)\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh ky}{\sinh k}\cos kx\,dk + \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh ky}{\sinh k}\sin kx\,dk \end{aligned}$$

The integrand of the first integral is an even function of k, and the integrand of the second integral is an odd function of k. Consequently, the first integral can be integrated from 0 to ∞ as long as we multiply it by 2, and the second integral is 0.

$$\begin{aligned} &= \frac{1}{2\pi}\cdot 2\int_{0}^{\infty} \frac{\sinh ky}{\sinh k}\cos kx\,dk \\ &= \frac{1}{\pi}\int_{0}^{\infty} \frac{\sinh ky}{\sinh k}\cos kx\,dk \end{aligned}$$

By the convolution theorem then,

$$u(x,y) = \int_{-\infty}^{\infty} f(s)\left[\frac{1}{\pi}\int_{0}^{\infty} \frac{\sinh ky}{\sinh k}\cos k(x-s)\,dk\right]ds.$$

Interchange the order of integration to get the answer at the back of the book.

$$u(x,y) = \int_{0}^{\infty}\int_{-\infty}^{\infty} f(s)\frac{1}{\pi}\frac{\sinh ky}{\sinh k}\cos(kx-ks)\,ds\,dk$$

Do note that the Fourier transform of $\sinh ky/\sinh k$ can be evaluated,

$$\mathcal{F}^{-1}\left\{\frac{\sinh ky}{\sinh k}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh ky}{\sinh k}e^{ikx}\,dk = \frac{1}{2}\frac{\sin \pi y}{\cos \pi y + \cosh \pi x},$$

and the solution to the PDE can be expressed as the following single integral.

$$u(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} f(s)\frac{\sin \pi y}{\cos \pi y + \cosh \pi(x-s)}\,ds$$