Exercise 3

Find $f(t)$ if its Laplace transform is $F(s) = 1/[s(s^2+1)]$.

Solution

To find $f(t)$ we have to take the inverse Laplace transform of $F(s)$.

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$$

Unfortunately, this function is not in the table of Laplace transforms. It can be written in terms of functions with known Laplace transforms, though, using partial fraction decomposition.
Looking at the function $F(s)$, we see that the denominator is a product of a linear factor s with a quadratic factor s^2+1.

$$F(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

Multiplying the left and right sides by the least common denominator, $s(s^2+1)$, to get rid of the fractions gives

$$1 = A(s^2+1) + s(Bs+C)$$

Plugging in $s = 0$ to this equation gives

$$1 = A(1) \quad \rightarrow \quad A = 1$$

Plugging in $s = 1$ and $s = 2$ gives a system of equations for B and C.

$$\begin{aligned} 1 &= 2 + B + C \\ 1 &= 5 + 2(2B + C) \end{aligned}$$

Solving this system gives $B = -1$ and $C = 0$. So

$$F(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

And

$$f(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{=1} - \underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}}_{=\cos t}$$

$$f(t) = 1 - \cos t$$