Exercise 5

Use the Laplace transform to solve $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, $u(0,t) = u(l,t) = 0$, $u(x,0) = \sin(\pi x/l)$, and $u_t(x,0) = -\sin(\pi x/l)$.

Solution

Let the Laplace transform of a function $u(x,t)$ be defined as

$$\bar{u}(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^\infty u(x,t)e^{-st}\,dt.$$

Applying the Laplace transform to both sides of the PDE gives

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} &= \mathcal{L}\left\{c^2\frac{\partial^2 u}{\partial x^2}\right\} \\
s^2\bar{u}(x,s) - su(x,0) - u_t(x,0) &= c^2\frac{d^2}{dx^2}\mathcal{L}\{u\} \\
s^2\bar{u} - s\sin\left(\frac{\pi x}{l}\right) - \left[-\sin\left(\frac{\pi x}{l}\right)\right] &= c^2\frac{d^2\bar{u}}{dx^2} \\
\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} &= \frac{1}{c^2}(1-s)\sin\frac{\pi x}{l}
\end{aligned}$$

What we have is an inhomogeneous ordinary differential equation. The general solution is therefore written as the sum of a complementary solution and a particular solution.

$$\bar{u} = \bar{u}_c + \bar{u}_p$$

The complementary solution is obtained from solving the associated homogeneous differential equation.

$$\frac{d^2\bar{u}_c}{dx^2} - \frac{s^2}{c^2}\bar{u}_c = 0$$

$$\bar{u}_c(x,s) = C_1\cosh\frac{s}{c}x + C_2\sinh\frac{s}{c}x$$

The constants, C_1 and C_2, are determined from the given boundary conditions of the problem.

$$\mathcal{L}\{u(0,t)\} = \bar{u}(0,s) = \mathcal{L}\{0\} = 0$$
$$\mathcal{L}\{u(l,t)\} = \bar{u}(l,s) = \mathcal{L}\{0\} = 0$$

$$\begin{aligned}
\bar{u}_c(0,s) &= C_1 = 0 & &\rightarrow \quad C_1 = 0 \\
\bar{u}_c(l,s) &= C_2\sinh\frac{s}{c}l = 0 & &\rightarrow \quad C_2 = 0
\end{aligned}$$

$$\bar{u}_c = 0$$

Because the right-hand side of the inhomogeneous differential equation is in terms of $\sin\frac{\pi x}{l}$, we can use the method of undetermined coefficients to find $\bar{u}_p$. We assume that $\bar{u}_p = A\cos\frac{\pi x}{l} + B\sin\frac{\pi x}{l}$, and we plug this into the equation to determine the coefficients.

$$-\frac{\pi^2}{l^2}A\cos\frac{\pi x}{l} - \frac{\pi^2}{l^2}B\sin\frac{\pi x}{l} - \frac{s^2}{c^2}A\cos\frac{\pi x}{l} - \frac{s^2}{c^2}B\sin\frac{\pi x}{l} = \frac{1}{c^2}(1-s)\sin\frac{\pi x}{l}$$
$$\left(-\frac{\pi^2}{l^2}A - \frac{s^2}{c^2}A\right)\cos\pi x + \left(-\frac{\pi^2}{l^2}B - \frac{s^2}{c^2}B\right)\sin\pi x = \frac{1}{c^2}(1-s)\sin\frac{\pi x}{l}$$

Matching coefficients on the left and right sides gives

$$-\frac{\pi^2}{l^2}A - \frac{s^2}{c^2}A = 0 \qquad \to \qquad A = 0$$
$$-\frac{\pi^2}{l^2}B - \frac{s^2}{c^2}B = \frac{1}{c^2}(1-s) \qquad \to \qquad B = \frac{l^2}{s^2l^2 + c^2\pi^2}(s-1).$$

So

$$\bar{u}_p = \frac{l^2}{s^2l^2 + c^2\pi^2}(s-1)\sin\frac{\pi x}{l}.$$

And the solution to the inhomogeneous differential equation is

$$\bar{u}(x,s) = \frac{l^2}{s^2l^2 + c^2\pi^2}(s-1)\sin\frac{\pi x}{l}.$$

All that's left to do now is to take the inverse Laplace transform to find $u(x,t)$.

$$u(x,t) = \mathcal{L}^{-1}\{\bar{u}(x,s)\}$$
$$u(x,t) = \mathcal{L}^{-1}\left\{\frac{l^2}{s^2l^2 + c^2\pi^2}(s-1)\sin\frac{\pi}{l}x\right\}$$
$$u(x,t) = \sin\frac{\pi x}{l}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \frac{c^2\pi^2}{l^2}}(s-1)\right\}$$
$$u(x,t) = \sin\frac{\pi x}{l}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \frac{c^2\pi^2}{l^2}} - \frac{1}{s^2 + \frac{c^2\pi^2}{l^2}}\right\}$$
$$u(x,t) = \sin\frac{\pi x}{l}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \frac{c^2\pi^2}{l^2}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\frac{c\pi}{l}}\frac{\frac{c\pi}{l}}{s^2 + \frac{c^2\pi^2}{l^2}}\right\}\right)$$
$$u(x,t) = \sin\frac{\pi x}{l}\left(\cos\frac{c\pi t}{l} - \frac{1}{\frac{c\pi}{l}}\sin\frac{c\pi t}{l}\right)$$

Therefore,

$$u(x,t) = \frac{1}{c\pi}\left(c\pi\cos\frac{c\pi t}{l} - l\sin\frac{c\pi t}{l}\right)\sin\frac{\pi x}{l}.$$

We can check to see whether this is the correct solution. Take derivatives of u with respect to x and t.

$$u_t = -\frac{1}{l}\left(l\cos\frac{c\pi t}{l} + c\pi\sin\frac{c\pi t}{l}\right)\sin\frac{\pi x}{l}$$
$$u_{tt} = \frac{c\pi}{l^2}\left(-c\pi\cos\frac{c\pi t}{l} + l\sin\frac{c\pi t}{l}\right)\sin\frac{\pi x}{l}$$
$$u_x = \frac{1}{cl}\left(c\pi\cos\frac{c\pi t}{l} - l\sin\frac{c\pi t}{l}\right)\cos\frac{\pi x}{l}$$
$$u_{xx} = \frac{\pi}{cl^2}\left(-c\pi\cos\frac{c\pi t}{l} + l\sin\frac{c\pi t}{l}\right)\sin\frac{\pi x}{l}$$

$u_{tt} = c^2u_{xx}$, so this is indeed the correct solution. By inspection we see that plugging $x = 0$ and $x = l$ into $u(x,t)$ gives $u = 0$. Also, plugging $t = 0$ into $u(x,t)$ and u_t gives $u = \sin(\pi x/l)$ and $u_t = -\sin(\pi x/l)$, respectively, so the initial and boundary conditions are satisfied.