Exercise 6

Use the Laplace transform to solve

$$u_{tt} = c^2 u_{xx} + \cos\omega t \sin\pi x \qquad \text{for } 0 < x < 1$$
$$u(0,t) = u(1,t) = u(x,0) = u_t(x,0) = 0.$$

Assume that $\omega > 0$ and be careful of the case $\omega = c\pi$. Check your answer by direct differentiation.

Solution

Let the Laplace transform of a function $u(x,t)$ be defined as

$$\bar{u}(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^\infty u(x,t)e^{-st}\,dt.$$

Applying the Laplace transform to both sides of the PDE gives

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} &= \mathcal{L}\left\{c^2\frac{\partial^2 u}{\partial x^2} + \cos\omega t\sin\pi x\right\} \\
s^2\bar{u}(x,s) - su(x,0) - u_t(x,0) &= c^2\frac{d^2}{dx^2}\mathcal{L}\{u\} + \mathcal{L}\{\cos\omega t\}\sin\pi x \\
s^2\bar{u}(x,s) &= c^2\frac{d^2\bar{u}}{dx^2} + \frac{s}{s^2+\omega^2}\sin\pi x \\
\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} &= -\frac{1}{c^2}\frac{s}{s^2+\omega^2}\sin\pi x
\end{aligned}$$

What we have is an inhomogeneous ordinary differential equation. The general solution is therefore written as the sum of a complementary solution and a particular solution.

$$\bar{u} = \bar{u}_c + \bar{u}_p$$

The complementary solution is obtained from solving the associated homogeneous differential equation.

$$\frac{d^2\bar{u}_c}{dx^2} - \frac{s^2}{c^2}\bar{u}_c = 0$$
$$\bar{u}_c(x,s) = C_1\cosh\frac{s}{c}x + C_2\sinh\frac{s}{c}x$$

The constants, C_1 and C_2, are determined from the given boundary conditions of the problem.

$$\mathcal{L}\{u(0,t)\} = \bar{u}(0,s) = \mathcal{L}\{0\} = 0$$
$$\mathcal{L}\{u(1,t)\} = \bar{u}(1,s) = \mathcal{L}\{0\} = 0$$

$$\bar{u}_c(0,s) = C_1 = 0 \qquad \rightarrow \qquad C_1 = 0$$
$$\bar{u}_c(1,s) = C_2\sinh\frac{s}{c} = 0 \qquad \rightarrow \qquad C_2 = 0$$

$$\bar{u}_c = 0$$

Because the right-hand side of the inhomogeneous differential equation is in terms of $\sin \pi x$, we can use the method of undetermined coefficients to find $\bar{u}_p$. We assume that $\bar{u}_p = A\cos \pi x + B \sin \pi x$, and we plug this into the equation to determine the coefficients.

$$\begin{gathered} -\pi^2 A \cos \pi x - \pi^2 B \sin \pi x - \frac{s^2}{c^2} A \cos \pi x - \frac{s^2}{c^2} B \sin \pi x = -\frac{1}{c^2}\frac{s}{s^2+\omega^2}\sin \pi x \\ \left(-\pi^2 A - \frac{s^2}{c^2}A\right)\cos \pi x + \left(-\pi^2 B - \frac{s^2}{c^2}B\right)\sin \pi x = -\frac{1}{c^2}\frac{s}{s^2+\omega^2}\sin \pi x \end{gathered}$$

Matching coefficients on the left and right sides gives

$$\begin{aligned} -\pi^2 A - \frac{s^2}{c^2}A &= 0 & &\rightarrow \quad A = 0 \\ -\pi^2 B - \frac{s^2}{c^2}B &= -\frac{1}{c^2}\frac{s}{s^2+\omega^2} & &\rightarrow \quad B = \frac{1}{s^2+c^2\pi^2}\cdot\frac{s}{s^2+\omega^2}. \end{aligned}$$

So

$$\bar{u}_p = \frac{1}{s^2+c^2\pi^2}\cdot\frac{s}{s^2+\omega^2}\sin \pi x.$$

And the solution to the inhomogeneous differential equation is

$$\bar{u}(x,s) = \frac{s\sin \pi x}{(s^2+\omega^2)(s^2+c^2\pi^2)}.$$

All that's left to do now is to take the inverse Laplace transform to find $u(x,t)$.

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}\{\bar{u}(x,s)\} \\ u(x,t) &= \mathcal{L}^{-1}\left\{\frac{s\sin \pi x}{(s^2+\omega^2)(s^2+c^2\pi^2)}\right\} \\ u(x,t) &= \sin \pi x\, \mathcal{L}^{-1}\left\{\frac{s}{(s^2+\omega^2)}\cdot\frac{1}{(s^2+c^2\pi^2)}\right\} \\ u(x,t) &= \sin \pi x\, \mathcal{L}^{-1}\left\{\frac{s}{(s^2+\omega^2)}\cdot\frac{1}{c\pi}\frac{c\pi}{(s^2+c^2\pi^2)}\right\} \\ u(x,t) &= \frac{\sin \pi x}{c\pi}\int_0^t \sin c\pi(t-t')\cos \omega t'\, dt' \end{aligned}$$

We have to be careful here since the value of the integral depends on whether $\omega = c\pi$ or not.

$$\int_0^t \sin c\pi(t-t')\cos \omega t'\, dt' = \begin{cases} \dfrac{c\pi}{\omega^2-c^2\pi^2}(\cos c\pi t - \cos \omega t) & \omega \neq c\pi \\ \dfrac{1}{2}t\sin c\pi t & \omega = c\pi \end{cases}$$

When $\omega = c\pi$, the phenomenon of resonance occurs, and the amplitude of the wave grows linearly with respect to time. Therefore,

$$u(x,t) = \begin{cases} \dfrac{\cos c\pi t - \cos \omega t}{\omega^2 - c^2\pi^2}\sin \pi x & \omega \neq c\pi \\ \dfrac{t\sin c\pi t}{2c\pi}\sin \pi x & \omega = c\pi \end{cases}.$$

We can check to see whether this is the correct solution. Take derivatives of u with respect to x and t.

$$u_t = \begin{cases} \dfrac{-c\pi \sin c\pi t + \omega \sin \omega t}{\omega^2 - c^2\pi^2} \sin \pi x & \omega \neq c\pi \\ \dfrac{c\pi t \cos c\pi t + \sin c\pi t}{2c\pi} \sin \pi x & \omega = c\pi \end{cases}$$

$$u_{tt} = \begin{cases} \dfrac{-c^2\pi^2 \cos c\pi t + \omega^2 \cos \omega t}{\omega^2 - c^2\pi^2} \sin \pi x & \omega \neq c\pi \\ \dfrac{2\cos c\pi t - c\pi t \sin c\pi t}{2} \sin \pi x & \omega = c\pi \end{cases}$$

$$u_x = \begin{cases} \pi \dfrac{\cos c\pi t - \cos \omega t}{\omega^2 - c^2\pi^2} \cos \pi x & \omega \neq c\pi \\ \pi \dfrac{t \sin c\pi t}{2c\pi} \cos \pi x & \omega = c\pi \end{cases}$$

$$u_{xx} = \begin{cases} -\pi^2 \dfrac{\cos c\pi t - \cos \omega t}{\omega^2 - c^2\pi^2} \sin \pi x & \omega \neq c\pi \\ -\pi^2 \dfrac{t \sin c\pi t}{2c\pi} \sin \pi x & \omega = c\pi \end{cases}$$

And so we have the following.

$$u_{tt} - c^2 u_{xx} = \begin{cases} \dfrac{\cancel{-c^2\pi^2 \cos c\pi t} + \cancel{c^2\pi^2 \cos c\pi t} + (\omega^2 - c^2\pi^2) \cos \omega t}{\omega^2 - c^2\pi^2} \sin \pi x = \cos \omega t \sin \pi x & \omega \neq c\pi \\ \dfrac{2\cos c\pi t - \cancel{c\pi t \sin c\pi t} + \cancel{c\pi t \sin c\pi t}}{2} \sin \pi x = \cos c\pi t \sin \pi x & \omega = c\pi \end{cases}$$

Thus, $u(x,t)$ satisfies the PDE. By inspection we can see that plugging in $t = 0$, $x = 0$, and $x = 1$ into $u(x,t)$ gives $u = 0$. Also, plugging in $t = 0$ into u_t above gives $u = 0$, so the initial and boundary conditions are satisfied.