Exercise 7

Use the Laplace transform to solve $u_t = ku_{xx}$ in $(0, l)$, with $u_x(0, t) = 0$, $u_x(l, t) = 0$, and $u(x, 0) = 1 + \cos(2\pi x/l)$.

Solution

Let the Laplace transform of a function $u(x, t)$ be defined as

$$\bar{u}(x, s) = \mathcal{L}\{u(x, t)\} = \int_0^\infty u(x, t)e^{-st}\, dt.$$

Applying the Laplace transform to both sides of the PDE gives

$$\begin{aligned}
\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} &= \mathcal{L}\left\{k\frac{\partial^2 u}{\partial x^2}\right\} \\
s\bar{u}(x, s) - u(x, 0) &= k\frac{d^2}{dx^2}\mathcal{L}\{u\} \\
s\bar{u}(x, s) - \left(1 + \cos\frac{2\pi x}{l}\right) &= k\frac{d^2\bar{u}}{dx^2} \\
\frac{d^2\bar{u}}{dx^2} - \frac{s}{k}\bar{u} &= -\frac{1}{k}\left(1 + \cos\frac{2\pi x}{l}\right)
\end{aligned}$$

What we have is an inhomogeneous ordinary differential equation. The general solution is therefore written as the sum of a complementary solution and a particular solution.

$$\bar{u} = \bar{u}_c + \bar{u}_p$$

The complementary solution is obtained from solving the associated homogeneous differential equation.

$$\frac{d^2\bar{u}_c}{dx^2} - \frac{s}{k}\bar{u}_c = 0$$

$$\bar{u}_c(x, s) = C_1 \cosh\sqrt{\frac{s}{k}}x + C_2 \sinh\sqrt{\frac{s}{k}}x$$

The constants, C_1 and C_2, are determined from the given boundary conditions of the problem.

$$\begin{aligned}
\mathcal{L}\{u_x(0, t)\} &= \bar{u}_x(0, s) = \mathcal{L}\{0\} = 0 \\
\mathcal{L}\{u_x(l, t)\} &= \bar{u}_x(l, s) = \mathcal{L}\{0\} = 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\bar{u}_c(x, s) &= C_1\sqrt{\frac{s}{k}} \sinh\sqrt{\frac{s}{k}}x + C_2\sqrt{\frac{s}{k}} \cosh\sqrt{\frac{s}{k}}x \\
\frac{d}{dx}\bar{u}_c(0, s) &= C_2\sqrt{\frac{s}{k}} = 0 \quad \rightarrow \quad C_2 = 0 \\
\frac{d}{dx}\bar{u}_c(l, s) &= C_1\sqrt{\frac{s}{k}} \sinh\sqrt{\frac{s}{k}}l = 0 \quad \rightarrow \quad C_1 = 0
\end{aligned}$$

$$\bar{u}_c = 0$$

The variation-of-parameters method can be used to find the particular solution. However, it leads to very complicated integrals. It's best to split up the particular solution into two parts. That is,

$$\bar{u}_p = \bar{u}_{p_1} + \bar{u}_{p_2},$$

where $\bar{u}_{p_1}$ and $\bar{u}_{p_2}$ satisfy

$$\frac{d^2\bar{u}_{p_1}}{dx^2} - \frac{s}{k}\bar{u}_{p_1} = -\frac{1}{k}$$

and

$$\frac{d^2\bar{u}_{p_2}}{dx^2} - \frac{s}{k}\bar{u}_{p_2} = -\frac{1}{k}\cos\frac{2\pi x}{l}$$

For the first equation, the right-hand side is just a constant, so $\bar{u}_{p_1}$ is a constant with respect to x. That means the second derivative is 0.

$$\begin{aligned} -\frac{s}{k}\bar{u}_{p_1} &= -\frac{1}{k} \\ \bar{u}_{p_1} &= \frac{1}{s} \end{aligned}$$

For the second equation, the right-hand side is a cosine function, so we use the method of undetermined coefficients to find $\bar{u}_{p_2}$. We assume that $\bar{u}_{p_2} = A\cos\frac{2\pi x}{l} + B\sin\frac{2\pi x}{l}$, and we plug this into the equation to determine the coefficients.

$$\begin{aligned} -\frac{4\pi^2}{l^2}A\cos\frac{2\pi x}{l} - \frac{4\pi^2}{l^2}B\sin\frac{2\pi x}{l} - \frac{s}{k}A\cos\frac{2\pi x}{l} - \frac{s}{k}B\sin\frac{2\pi x}{l} &= -\frac{1}{k}\cos\frac{2\pi x}{l} \\ \left(-\frac{4\pi^2}{l^2}A - \frac{s}{k}A\right)\cos\frac{2\pi x}{l} + \left(-\frac{4\pi^2}{l^2}B - \frac{s}{k}B\right)\sin\frac{2\pi x}{l} &= -\frac{1}{k}\cos\frac{2\pi x}{l} \end{aligned}$$

Matching coefficients on the left and right sides gives

$$\begin{aligned} -\frac{4\pi^2}{l^2}A - \frac{s}{k}A = -\frac{1}{k} \qquad &\to \quad A = \frac{l^2}{4\pi^2 k + l^2 s} \\ -\frac{4\pi^2}{l^2}B - \frac{s}{k}B = 0 \qquad &\to \quad B = 0. \end{aligned}$$

So

$$\bar{u}_{p_2} = \frac{l^2}{4\pi^2 k + l^2 s}\cos\frac{2\pi x}{l}.$$

Therefore,

$$\bar{u}_p = \bar{u}_{p_1} + \bar{u}_{p_2} = \frac{1}{s} + \frac{l^2}{4\pi^2 k + l^2 s}\cos\frac{2\pi x}{l}.$$

And the solution to the inhomogeneous differential equation is

$$\bar{u}(x,s) = \frac{1}{s} + \frac{l^2}{4\pi^2 k + l^2 s}\cos\frac{2\pi x}{l}.$$

All that's left to do now is to take the inverse Laplace transform to find $u(x,t)$.

$$\begin{aligned}
u(x,t) &= \mathcal{L}^{-1}\{\bar{u}(x,s)\} \\
u(x,t) &= \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{l^2}{4\pi^2 k + l^2 s}\cos\frac{2\pi x}{l}\right\} \\
u(x,t) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{l^2}{4\pi^2 k + l^2 s}\cos\frac{2\pi x}{l}\right\} \\
u(x,t) &= \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{=1} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s + \frac{4\pi^2 k}{l^2}}\right\}}_{=e^{-\frac{4\pi^2 k}{l^2}t}}\cos\frac{2\pi x}{l}
\end{aligned}$$

Therefore,

$$u(x,t) = 1 + e^{-\frac{4\pi^2 k}{l^2}t}\cos\frac{2\pi x}{l}.$$

We can check to see whether this is the correct solution. Take derivatives of u with respect to x and t.

$$\begin{aligned}
u_t &= -\frac{4\pi^2 k}{l^2}e^{-\frac{4\pi^2 k}{l^2}t}\cos\frac{2\pi x}{l} \\
u_x &= -\frac{2\pi}{l}e^{-\frac{4\pi^2 k}{l^2}t}\sin\frac{2\pi x}{l} \\
u_{xx} &= -\frac{4\pi^2}{l^2}e^{-\frac{4\pi^2 k}{l^2}t}\cos\frac{2\pi x}{l}
\end{aligned}$$

$u_t = k u_{xx}$, so this is indeed the correct solution. By inspection we see that

$$u(x,0) = 1 + \cos\frac{2\pi x}{l}$$

and that plugging in $x = 0$ and $x = l$ into u_x above gives $u_x = 0$. Thus, the initial and boundary conditions are satisfied.